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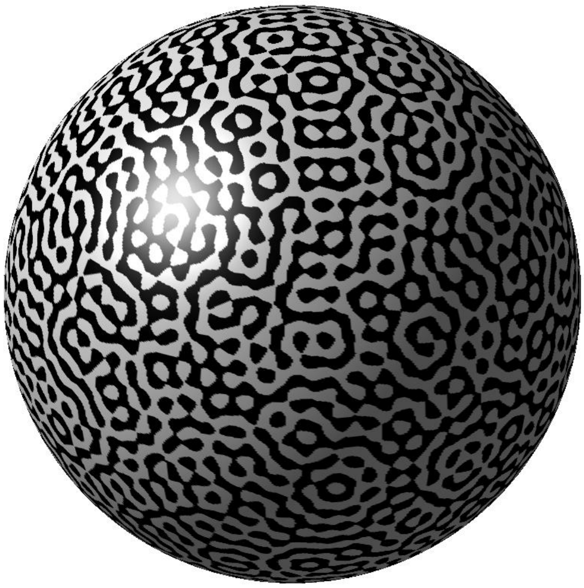
Contributions à l'étude des sous-variétés aléatoires



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Thèse de doctorat

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[1] à temps partiel, certes

[2] Je persiste, c'est saumon.

[3] principalement

[4] Sers toi de ta tête !

pertinentes, des discussions vraiment pas pertinentes[5], des séances d'escalade, des footings, une Sainté-Lyon, des randonnées, de la musique, des hostilités ou du jus de houblon. À savoir : Alejandro, Alexandre, Ariane, Arthur, Benoît, Bruno, Cécile F.[6], Cécilia, Colin, Coline, Corentin et son air fâché, Cyrielle, François L.M.[7] et son plan, Fred, Grégoire, Hugo[8] le nouveau maître du *Dixit*, Isa, Ivan, Julien, Kévin[9], Léo, Loïc, Mathias[10], Mickaël, Mini-Bloup[11], Nadja, Niccolò, le Nunz'[12], Romain[13], Samuel, Simon, Sylvain[14], Valentin, Xiaolin, Yohann et tous ceux que j'ai probablement oublié.

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[5]Si si, rappelez-vous.
[6]Le chœur c'est bien, mais ...
[7]Vas-y champion !
[8]stagiaire bio local
[9]cordialement
[10]90♠ !
[11]Wesh gros !
[12]cordialement aussi
[13]La 6a !
[14]C'est pas chez nous !

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CHAPITRE 1

Introduction

L'objet de cette thèse est l'étude statistique de certaines propriétés, géométriques ou topologiques, de sous-variétés aléatoires dans une variété compacte. On peut faire remonter l'origine de ce problème à l'article de M. Kac [Kac43], paru en 1943. Le résultat de cet article nous sert d'exemple introductif et est présenté ci-dessous (section 1.1.1).

Dans les sections 1.1.2 et 1.1.3 nous présentons notre cadre de travail général et nous en discutons succintement les hypothèses. Nous présentons ensuite (sections 1.2 et 1.3) les deux modèles de sous-variétés aléatoires que nous considérons et les résultats que nous obtenons pour chacun de ces modèles. La section 1.2 est consacrée à un modèle riemannien et la section 1.3 à un modèle algébrique réel. Dans la section 1.4 nous donnons une idée des démonstrations de nos théorèmes principaux. Enfin, la section 1.5 détaille l'organisation du reste de ce manuscrit.

Nos résultats principaux sont les théorèmes 1.2.5, 1.2.6, 1.3.13, 1.3.14 et 1.3.19.

1.1 Généralités

1.1.1 Polynômes de Kac

Considérons un polynôme P de degré d à coefficients complexes. On sait que P possède d racines complexes, qui sont génériquement distinctes. Posons maintenant cette question dans le monde réel. Étant donné un polynôme P de degré d à coefficients réels, combien a-t-il de racines réelles ? La réponse est tout de suite beaucoup moins claire. On sait que le nombre de racines de P est compris entre 0 et d et qu'il est génériquement de la même parité que d, mais il n'est pas facile à déterminer. Comme souvent lorsqu'un problème déterministe est trop complexe, on peut introduire de l'aléa et chercher un comportement typique au sens de la mesure. C'est ce que fait Kac dans [Kac43], où il obtient le résultat suivant.

Théorème 1.1.1 (Kac, 1943). *Soit $d \in \mathbb{N}^*$ et soit $P = \sum_{i=0}^{d} a_i X^i$, où $a_0, \dots, a_d$ sont des variables aléatoires indépendantes identiquement distribuées de loi normale centrée réduite, alors le nombre moyen de racines distinctes de P dans l'intervalle ouvert $I \subset \mathbb{R}$ est :*

$$\mathbb{E}\left[\operatorname{card}\left(P^{-1}(0) \cap I\right)\right] = \frac{1}{\pi} \int_I \frac{\left(t^{4d} - d^2 t^{2(d+1)} + 2(d^2-1)t^{2d} - d^2 t^{2(d-1)} + 1\right)^{\frac{1}{2}}}{(1-t^2)(1-t^{2d})} \,\mathrm{d}t.$$

De plus, $\mathbb{E}\left[\operatorname{card}\left(P^{-1}(0)\right)\right] \sim \dfrac{2}{\pi}\ln d$ *lorsque* $d \to +\infty$.

En dimension supérieure, on peut s'intérroger sur le volume typique d'une hypersurface algébrique réelle. Cela a peu de sens pour une hypersurface algébrique de $\mathbb{R}^n$ car son volume est souvent infini. Sans surprise, le bon cadre pour se poser cette question est celui des hypersurfaces algébriques de l'espace projectif réel $\mathbb{RP}^n$. On peut reformuler le résultat de Kac dans ce cadre, en homogénéisant le polynôme P et en voyant $\mathbb{R}$ comme une carte affine de $\mathbb{RP}^1$.

1.1.2 Cadre d'étude

On peut poser des questions similaires dans le cadre général suivant. Soit M une variété ambiante de dimension n, que l'on suppose fermée (c'est-à-dire compacte sans bord) et lisse (c'est-à-dire de classe $\mathcal{C}^\infty$). Soit V un sous-espace de l'espace $\mathcal{C}^\infty(M)$ des fonctions lisses sur M et soit $\mathrm{d}\nu$ une mesure de probabilité sur V. Soit enfin $f \in V$ une fonction aléatoire distribuée selon $\mathrm{d}\nu$. Sous certaines hypothèses d'amplitude précisées plus bas, le lieu Z_f des zéros de f est presque surement une hypersurface lisse de M. Dans ce cas, que peut-on dire de la topologie de Z_f, ou de sa géométrie si M est équipée d'une métrique riemannienne ? On cherche évidemment un réponse statistique à ces questions.

Présenté ainsi, le problème est très vaste et dépend fortement des choix de V et $\mathrm{d}\nu$. Dans ce manuscrit, nous présentons des résultats portant sur l'espérance du volume et de la caractéristique d'Euler d'hypersurfaces du type précédent dans deux cas particuliers : un cadre riemannien, décrit à la section 1.2, et un cadre algébrique réel, décrit à la section 1.3. Dans le second cas, nous obtenons aussi un résultat portant sur la variance du volume de Z_f. Nous considérons également des sous-variétés aléatoires de codimension r, avec $r \in \{1, \dots, n\}$, obtenues comme lieu d'annulation d'applications aléatoires $f : M \to \mathbb{R}^r$ définies par un procédé similaire. Notons que, dans le cadre algébrique réel, il faut remplacer V dans la discussion précédente par un espace de sections lisses d'un certain fibré vectoriel de rang r au-dessus de M.

Soient $r \in \{1, \dots, n\}$ et V un espace d'applications lisses de M dans $\mathbb{R}^r$, nous présentons maintenant des conditions suffisantes non nécessaires sur V et la mesure de probabilité $\mathrm{d}\nu$ pour que, $\mathrm{d}\nu$-presque surement, Z_f soit une sous-variété lisse de codimension r de M. Ces hypothèses seront vérifiées dans nos cadres de travail riemannien et algébrique réel.

Définition 1.1.2. On dit que l'espace V est 0-*ample* si, pour tout $x \in M$, l'application d'évaluation :

$$\begin{array}{rccc} \mathrm{ev}_x : & V & \longrightarrow & \mathbb{R}^r \\ & f & \longmapsto & f(x) \end{array}$$

est surjective.

Nous reprenons ici la terminologie de l'article [Nic15c] dans lequel Nicolaescu définit plus généralement la k-amplitude comme la surjectivité en tout point de l'évaluation du k-jet. La notion de 0-amplitude s'adapte immédiatement au cas où V est un espace de sections d'un fibré vectoriel au-dessus de M.

Si l'on suppose que V est 0-ample et de dimension finie alors, par une application du théorème de Sard, l'ensemble des fonctions $f \in V$ qui ne s'annulent pas transversalement est négligeable pour la mesure de Lebesgue sur V. En particulier, pour f dans le complémentaire de cet ensemble négligeable, Z_f est bien une sous-variété de codimension r. Si de plus $\mathrm{d}\nu$ est à densité par rapport à la mesure de Lebesgue, alors Z_f est $\mathrm{d}\nu$-presque surement une sous-variété de codimension r. Cet argument est détaillé plus bas, section 2.2.2 dans le cas de fonctions à valeurs dans $\mathbb{R}^r$, et section 2.2.6 dans le cas de sections d'un fibré.

1.1.3 Choix de la mesure de probabilité

Dans cette section, nous supposons V de dimension finie et 0-ample et nous discutons le choix de la mesure $d\nu$ sur V. Avant cela, on rappelle la définition d'un vecteur gaussien dans le cas non dégénéré, qui est celui que nous rencontrerons.

Définition 1.1.3. Soient $(V, \langle \cdot \,, \cdot \rangle)$ un espace euclidien de dimension N, $m \in V$ et Λ un opérateur auto-adjoint défini positif sur V, la loi *gaussienne* (ou *normale*) de *moyenne* m et de *variance* Λ est la mesure de probabilité admettant la densité :

$$x \longmapsto \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det(\Lambda)}} \exp\left(-\frac{1}{2}\left\langle \Lambda^{-1}(x-m)\,, (x-m)\right\rangle\right)$$

par rapport à la mesure de Lebesgue sur V, normalisée pour qu'un cube unité soit de volume 1. On dit que la loi est *centrée* si $m = 0$ et *réduite* si $\Lambda = \mathrm{Id}$. On parle de gaussienne *standard* pour désigner la distribution gaussienne centrée réduite.

Notation 1.1.4. On note $X \sim \mathcal{N}(m, \Lambda)$ le fait que le vecteur aléatoire X soit distribué selon la loi normale de moyenne m et de variance Λ. Lorsque $m = 0$, comme ce sera systématiquement le cas dans la suite, on note parfois $X \sim \mathcal{N}(\Lambda)$.

Si $(e_1, \dots, e_n)$ est une base orthonormée de V, un vecteur aléatoire $X \in V$ est gaussien standard si et seulement si il est de la forme :

$$X = \sum_{i=1}^{n} a_i e_i,$$

où $a_1, \dots, a_n$ sont des variables aléatoires réelles indépendantes identiquement distribuées de loi gaussienne standard.

Revenons au choix de la mesure de probabilité $d\nu$ sur V qui va nous permettre de définir nos sous-variétés aléatoires. Dans les cas qui nous intéressent, V sera muni d'un produit scalaire naturel. Supposons donc désormais que V est euclidien. Le choix du produit scalaire sur V n'est pas anodin, mais nous supposons ici que ce choix a été fait.

Avec les mêmes notations qu'à la section précédente, pour que Z_f soit presque surement une sous-variété de la bonne dimension, il faut en particulier que $d\nu$ ne charge pas 0. Ensuite, étant donné que deux fonctions non nulles et colinéaires s'annulent sur le même ensemble, l'ensemble $\{Z_f \mid f \in V \setminus \{0\}\}$ est en fait paramétré par l'espace projectif $\mathbb{P}(V)$. Si $d\nu$ est une mesure de probabilité sur V qui ne charge pas 0, la distribution de Z_f ne dépend alors que de la mesure de probabilité induite par $d\nu$ sur $\mathbb{P}(V)$. Un choix naturel est de demander que cette mesure induite soit uniforme, i.e. invariante sous l'action du groupe orthogonal de V. Ceci étant fixé, on peut choisir pour $d\nu$ n'importe quelle mesure de probabilité induisant la mesure de probabilité uniforme sur $\mathbb{P}(V)$. Ce choix n'a pas d'influence sur la distribution de Z_f. On dispose, là encore, d'un candidat naturel : la mesure gaussienne centrée réduite sur V. On peut aussi considérer une gaussienne centrée dont la variance est une homothétie.

En conclusion, pour obtenir un modèle raisonnable de sous-variétés aléatoires lisses de codimension r, il nous suffit de choisir un espace euclidien de fonctions lisses de M dans $\mathbb{R}^r$ (ou de sections d'un fibré de rang r) qui soit 0-ample. Le choix de la mesure gaussienne standard sur cet espace est ensuite relativement naturel. Comme cette mesure est à densité par rapport à la mesure de Lebesgue, la 0-amplitude de V garantit que Z_f sera presque surement lisse et de la dimension attendue.

L'utilisation de distributions gaussiennes présente plusieurs avantages. Premièrement, ces lois sont totalement déterminées par leurs premier et second moments. Notamment, deux vecteurs aléatoires dont la loi jointe est gaussienne sont indépendants si et seulement si ils sont décorrélés, ce qui se lit sur l'opérateur de variance. La notion d'indépendance est donc particulièrement simple dans le monde gaussien. Ensuite, les vecteurs gaussiens se comportent agréablement vis-à-vis des applications linéaires et du conditionnement. L'image d'un vecteur gaussien $X \sim \mathcal{N}(m, \Lambda)$ par une application linéaire L entre espaces euclidiens est un vecteur gaussien de moyenne $L(m)$ et de variance $L\Lambda L^*$, où L^* est l'adjoint de L. De même, si (X, Y) est un vecteur gaussien, la loi conditionnelle de Y sachant que $X = x$ est encore gaussienne, et sa moyenne et sa variance s'expriment simplement à partir de celles de (X, Y) et de la valeur de x (voir proposition 2.A.10). Ces faits seront abondamment utilisés dans la suite. Les définitions et propriétés utiles des vecteurs gaussiens sont rappelées dans l'appendice 2.A.

1.2 Cadre riemannien

Dans cette section nous décrivons un modèle de sous-variétés aléatoires d'une variété riemannienne, définies comme le lieu d'annulation de combinaisons linéaires aléatoires de fonctions propres du laplacien (section 1.2.1). Nous énonçons ensuite les résultats que nous obtenons dans ce cadre dans la section 1.2.2. La section 1.2.3 présente deux autres modèles de sous-variétés aléatoires définies à partir des fonctions propres du laplacien et rappelle un certain nombre de résultats concernant ces modèles.

1.2.1 Ondes riemanniennes aléatoires

Soit (M, g) une variété riemannienne fermée de dimension n. La métrique g permet de définir une mesure de volume $|\mathrm{d}V_M|$ sur M de la façon suivante. Si ϕ est une fonction à support dans un ouvert de carte U et $(x_1, \dots, x_n)$ sont des coordonnées sur U, alors

$$\int_M \phi \, |\mathrm{d}V_M| = \int \phi(x) \sqrt{\det(g_{ij}(x))} \, \mathrm{d}x_1 \dots \mathrm{d}x_n, \tag{1.2.1}$$

où $x = (x_1, \dots, x_n)$ et $(g_{ij}(x))$ est la matrice décrivant la métrique g au point x dans nos coordonnées locales. Cette mesure riemannienne permet de définir un produit scalaire euclidien L^2 sur $\mathcal{C}^\infty(M)$. Pour tout f_1 et $f_2 \in \mathcal{C}^\infty(M)$, on pose :

$$\langle f_1 \, , f_2 \rangle = \int_{x \in M} f_1(x) f_2(x) \, |\mathrm{d}V_M| \, . \tag{1.2.2}$$

Soit maintenant Δ l'opérateur de Laplace–Beltrami de $\mathcal{C}^\infty(M)$ dans lui-même. On rappelle que $\Delta = d^*d$ où $d : \mathcal{C}^\infty(M) \to \Omega^1(M)$ est la différentielle usuelle et d^* est son adjoint formel pour les produits scalaires L^2 induits par g. On a alors le résultat classique suivant (voir par exemple [GHL04, thm. 4.43]).

Théorème 1.2.1.
- *On peut arranger les valeurs propres distinctes de Δ en une suite strictement croissante $(\lambda_k)_{k \in \mathbb{N}}$ qui tend vers l'infini :*

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \cdots$$

- *Pour tout $k \in \mathbb{N}$, l'espace propre* $\ker(\Delta - \lambda_k \operatorname{Id})$ *est de dimension finie.*
- *La somme directe*

$$\bigoplus_{k \in \mathbb{N}} \ker(\Delta - \lambda_k \operatorname{Id})$$

est dense dans $\mathcal{C}^\infty(M)$ pour la topologie de la convergence uniforme et dans $L^2(M)$ pour la norme induite par le produit scalaire (1.2.2).

Soit $\lambda \geqslant 0$, on note $V_\lambda = \bigoplus_{\lambda_k \leqslant \lambda} \ker(\Delta - \lambda_k \operatorname{Id})$. Par le théorème précédent, cet espace est de dimension finie pour tout λ et le produit scalaire (1.2.2) en fait un espace euclidien. Pour tout $\lambda \geqslant 0$, V_λ contient les fonctions constantes non nulles, qui sont les fonctions propres de Δ associées à la valeur propre $\lambda_0 = 0$. En particulier, V_λ est toujours 0-ample.

Soit $f \sim \mathcal{N}(0, \operatorname{Id})$ dans V_λ, d'après la section précédente, le lieu d'annulation Z_f de f est presque surement une hypersurface lisse de M. C'est cette hypersurface aléatoire que nous étudions dans notre modèle riemannien. La fonction f est appelée *onde riemannienne aléatoire* (Riemannian random wave) et on parle d'hypersurface *nodale* aléatoire pour désigner Z_f.

Soit $r \in \{1, \dots, n\}$ et soient $f^{(1)}, \dots, f^{(r)} \in V_\lambda$ des vecteurs gaussiens standards indépendants. On note $f = (f^{(1)}, \dots, f^{(r)}) \in (V_\lambda)^r$, où $(V_\lambda)^r$ est muni du produit scalaire induit par celui de V_λ. En d'autres termes, on voit $(V_\lambda)^r$ comme un sous-espace de l'espace des applications lisses de M dans $\mathbb{R}^r$, muni du produit scalaire L^2 défini par :

$$\langle f_1 \,, f_2 \rangle = \int_{x \in M} \langle f_1(x) \,, f_2(x) \rangle \left| \mathrm{d}V_M \right|, \tag{1.2.3}$$

où le produit scalaire sous l'intégrale est le produit scalaire usuel de $\mathbb{R}^r$. Alors f est un vecteur gaussien standard dans $(V_\lambda)^r$. De plus, le lemme 2.2.7 montre que $(V_\lambda)^r$ est 0-ample pour tout $\lambda \geqslant 0$. On a donc que

$$Z_f = Z_{f^{(1)}} \cap \dots \cap Z_{f^{(r)}}$$

est presque surement une sous-variété de codimension r de M. Ceci définit les sous-variétés aléatoires de codimension r dans notre modèle d'ondes riemanniennes aléatoires.

Remarque 1.2.2. Le fait que $(V_\lambda)^r$ soit 0-ample implique en particulier que les hypersurfaces $Z_{f^{(i)}}$ s'intersectent transversalement, presque surement.

Pour fixer les idées, donnons l'exemple le plus simple du modèle que nous venons de décrire. On considère le cercle euclidien $\mathbb{S}^1$, vu comme $\mathbb{R}/2\pi\mathbb{Z}$, avec une coordonnée angulaire notée θ. L'opérateur de Laplace–Beltrami est dans ce cas $\Delta = -\frac{\partial^2}{\partial \theta^2}$. Les valeurs propres de Δ sont les $\{k^2 \mid k \in \mathbb{N}\}$, et pour $k \in \mathbb{N}^*$ l'espace propre associé à k^2 est de dimension 2, engendré par $\theta \mapsto \sin(k\theta)$ et $\theta \mapsto \cos(k\theta)$.

Dans cet exemple, l'espace V_λ est donc l'espace des polynômes trigonométriques de degré au plus $\lfloor \sqrt{\lambda} \rfloor$. On dispose aussi d'une base sympathique dans laquelle exprimer une fonction de V_λ. Soit $f \in V_\lambda$ une fonction aléatoire de loi $\mathcal{N}(0, \operatorname{Id})$, alors on peut écrire f sous la forme :

$$f : \theta \longmapsto \frac{1}{\sqrt{\pi}} \left(\frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\lfloor \sqrt{\lambda} \rfloor} a_k \cos(k\theta) + b_k \sin(k\theta) \right), \tag{1.2.4}$$

où les a_k et b_k sont des gaussiennes standards indépendantes à valeurs dans $\mathbb{R}$. Comme on ne s'intéresse qu'au lieu d'annulation de f, on peut se débarrasser du facteur $\frac{1}{\sqrt{\pi}}$ dans cette expression.

1.2.2 Énoncé des résultats

Le premier résultat concernant la géométrie des hypersurfaces nodales a été obtenu par Bérard [Bé85]. Il donne l'asymptotique de leur volume moyen dans le modèle d'ondes riemanniennes aléatoires que nous venons de décrire.

Théorème 1.2.3 (Bérard, 1985)**.** *Soit (M, g) une variété riemannienne fermée de dimension n et soit $f \in V_\lambda$ un vecteur gaussien standard, on a :*

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \sqrt{\frac{\lambda}{n+2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-1})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O(1)\,,$$

lorsque $\lambda \to +\infty$. Ici les volumes de M et Z_f sont les volumes riemanniens induits par g et les volumes des sphères sont les volumes euclidiens.

Ce résultat a ensuite été précisé par Zelditch [Zel09] qui prouve un résultat d'équidistribution en moyenne pour le lieu des zéros de f. Introduisons quelques notations supplémentaires pour énoncer ce résultat. Soit $f \in \mathcal{C}^\infty(M)$ telle que Z_f soit une hypersurface lisse, on note $|\mathrm{d}V_f|$ la mesure riemannienne d'intégration sur Z_f. Cette mesure est définie de façon similaire à (1.2.1) pour la métrique induite par la métrique ambiante. On peut ensuite voir $|\mathrm{d}V_f|$ comme une forme linéaire sur $\mathcal{C}^0(M)$, on note alors :

$$\langle |\mathrm{d}V_f|\,, \phi\rangle = \int_{Z_f} \phi\,|\mathrm{d}V_f|\,,$$

pour toute fonction-test ϕ.

Théorème 1.2.4 (Zelditch, 2009)**.** *Soit (M, g) une variété riemannienne fermée de dimension n et soit $f \in V_\lambda$ un vecteur gaussien standard, pour tout $\phi \in \mathcal{C}^0(M)$, on a :*

$$\mathbb{E}[\langle |\mathrm{d}V_f|\,, \phi\rangle] = \sqrt{\frac{\lambda}{n+2}}\left(\int_M \phi\,|\mathrm{d}V_M|\right)\frac{\mathrm{Vol}\,(\mathbb{S}^{n-1})}{\mathrm{Vol}\,(\mathbb{S}^n)} + o\!\left(\sqrt{\lambda}\right),$$

lorsque $\lambda \to +\infty$. En d'autres termes, on a la convergence suivante au sens faible entre mesures de Radon positives :

$$\sqrt{\frac{n+2}{\lambda}}\mathbb{E}[|\mathrm{d}V_f|] \xrightarrow[\lambda\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-1})}{\mathrm{Vol}\,(\mathbb{S}^n)}\,|\mathrm{d}V_M|\,.$$

On retrouve le résultat de Bérard en appliquant la première formulation ci-dessus à la fonction constante $\mathbf{1} : x \mapsto 1$. Notre première contribution est d'étendre ce résultat en codimension supérieure.

Théorème 1.2.5. *Soit (M, g) une variété riemannienne fermée de dimension n, soit $r \in \{1, \dots, n\}$ et soient $f^{(1)}, \dots, f^{(r)} \in V_\lambda$ des vecteurs gaussiens standards indépendants. On note Z_f le lieu des zéros communs de ces r fonctions et $|\mathrm{d}V_f|$ la mesure d'intégration sur Z_f. Alors, pour tout $\phi \in \mathcal{C}^0(M)$, on a :*

$$\mathbb{E}[\langle |\mathrm{d}V_f|\,, \phi\rangle] = \left(\frac{\lambda}{n+2}\right)^{\frac{r}{2}}\left(\int_M \phi\,|\mathrm{d}V_M|\right)\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_\infty\, O\!\left(\lambda^{\frac{r-1}{2}}\right),$$

lorsque $\lambda \to +\infty$. De plus le terme $O\!\left(\lambda^{\frac{r-1}{2}}\right)$ est indépendant de ϕ.

Dans cet énoncé, on a noté $\|\phi\|_\infty = \max_{x\in M} |\phi(x)|$. Cela montre en particulier que

$$\left(\frac{n+2}{\lambda}\right)^{\frac{r}{2}}\mathbb{E}[|\mathrm{d}V_f|] \xrightarrow[\lambda\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\,|\mathrm{d}V_M|\,,$$

en tant que formes linéaires continues sur $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$. La preuve de ce résultat est donné dans la section 2.5.2 dans le cas où la fonction-test est $\mathbf{1}$, c'est-à-dire dans le cas où on calcule le volume moyen de Z_f. L'ajout d'une fonction-test ne modifie en rien la preuve (voir la remarque 2.5.5) et la forme plus précise énoncée ici découle immédiatement de la formule de Kac–Rice (thm. 2.5.3) et du lemme 2.5.4.

Notre second résultat pour ce modèle est le calcul d'une asymptotique similaire pour la caractéristique d'Euler moyenne de sous-variétés nodales aléatoires. Dans la suite, on notera $\chi(Z_f)$ la caractéristique d'Euler de Z_f. Rappelons que la caractéristique d'Euler d'une variété fermée de dimension impaire est nulle. Si $n-r$ est impair, on a donc $\chi(Z_f) = 0$ presque surement.

Théorème 1.2.6. *Soit (M, g) une variété riemannienne fermée de dimension n, soit $r \in \{1, \ldots, n\}$ et soient $f^{(1)}, \ldots, f^{(r)} \in V_\lambda$ des vecteurs gaussiens standards indépendants. On note Z_f le lieu des zéros communs de ces r fonctions. Si $n - r$ est pair, on a l'asymptotique suivante quand λ tend vers l'infini :*

$$\mathbb{E}[\chi(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{\lambda}{n+2}\right)^{\frac{n}{2}} \mathrm{Vol}\,(M)\, \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right) \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \,\mathrm{Vol}\left(\mathbb{S}^{n}\right) \mathrm{Vol}\left(\mathbb{S}^{n-1}\right)} + O\Big(\lambda^{\frac{n-1}{2}}\Big).$$

La démonstration de ce théorème fait l'objet de la section 2.5.4.

1.2.3 Résultats liés

Commençons par introduire deux autres modèles de sous-variétés aléatoires proches des ondes riemanniennes aléatoires. À notre connaissance, tous les résultats pour ces modèles concernent des hypersurfaces et nous nous limitons donc à ce cadre.

Sur une variété riemannienne générique, les espaces propres du laplacien sont tous de dimension 1 (voir [Uhl76]). En revanche, dans certains cas particuliers, comme la sphère euclidienne $\mathbb{S}^n$ ou le tore plat $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, les espaces propres sont de grande dimension. Cela a alors du sens de considérer le lieu des zéros d'une fonction aléatoire $f \sim \mathcal{N}(0, \mathrm{Id})$ dans $\ker(\Delta - \lambda_k \,\mathrm{Id})$, et non plus dans la somme directe de ces espaces. On parle alors d'onde aléatoire *monochromatique* (monochromatic random wave). Dans le cas du tore, certains auteurs parlent d'onde aléatoire *arithmétique*, en référence aux problèmes de théorie des nombres qui surgissent quand on s'intéresse au spectre du laplacien sur $\mathbb{T}^n$. Les résultats dans ce cadre sont des résultats asymptotiques lorsque $k \to +\infty$.

Pour mémoire (cf. [BGM71, chap. 3] par exemple), sur la sphère euclidienne $\mathbb{S}^n$, les valeurs propres distinctes de Δ sont les $\lambda_k = k(k + n - 1)$ pour $k \in \mathbb{N}$. De plus, l'espace propre associé à λ_k est formé des restrictions à $\mathbb{S}^n$ des polynômes homogènes de degré k en $(n + 1)$ variables qui sont harmoniques dans $\mathbb{R}^{n+1}$. Cet espace est de dimension :

$$\binom{n+k}{k} - \binom{n+k-2}{k-2} = \frac{n+2k-1}{n+k-1}\binom{n+k-1}{k}.$$

Sur le tore plat $\mathbb{T}^n$, les valeurs propres de Δ sont les entiers de la forme $\|p\|^2$, où $\|\cdot\|$ désigne la norme usuelle sur $\mathbb{R}^n$ et $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. L'espace propre $\ker(\Delta - \lambda\,\mathrm{Id})$ est alors engendré par les fonctions :

$$x \mapsto \cos(\langle p\,, x\rangle) \qquad \text{et} \qquad x \mapsto \sin(\langle p\,, x\rangle)\,,$$

avec $p \in \mathbb{Z}^n$ tel que $\|p\|^2 = \lambda$.

Le troisième modèle riemannien que nous évoquerons est appelé *modèle de bande passante* dans [Ana16] (Band-limited random waves en anglais). Il consiste à étudier, dans une variété riemannienne fermée (M, g), les hypersurfaces nodales d'une fonction gaussienne standard dans l'espace

$$V_\lambda^\ell = \bigoplus_{\lambda_k \in [\ell\lambda, \lambda]} \ker(\Delta - \lambda_k\,\mathrm{Id})\,,$$

où $\ell \in [0,1[$. On dit que ℓ est la *largeur* de la bande passante. Notons que ce modèle contient celui des ondes riemanniennes aléatoires, pour $\ell = 0$.

Il existe d'autres modèles de sous-variétés aléatoires définies à partir de fonctions propres du laplacien. En plus des variations sur les trois modèles précédents, citons le modèle du champ libre gaussien tronqué (cut-off Gaussian free field) étudié, par exemple, dans [Riv16].

Dans [Zel09], Zelditch prouve le théorème 1.2.4 dans le cas des ondes riemanniennes aléatoires et dans un modèle de bande passante où il considère une bande de fréquence $[\lambda, \lambda+1]$. Dans ces deux cas, il montre également que pour toute fonction-test $\phi \in \mathcal{C}^0(M)$ $\frac{1}{\lambda}\operatorname{Var}(\langle |\mathrm{d}V_f|\,, \phi\rangle)$ est bornée, sous l'hypothèse supplémentaire que (M,g) soit analytique.

Dans [GW11b], Granville et Wigman étudient l'asymptotique du nombre de racines pour un polynôme trigonométrique aléatoire dont la distribution est très proche de (1.2.4) (c'est la même distribution où le terme de degré 0 a été supprimé). Ils calculent les asymptotiques de l'espérance et de la variance pour ce nombre de racines et démontrent qu'il vérifie un théorème central limite quand le degré du polynôme tend vers l'infini.

Les domaines nodaux d'ondes arithmétiques aléatoires ont été étudiées successivement dans [ORW08, RW08, KKW13] et [MPRW16]. L'article [ORW08] est consacré à l'étude de la mesure de Leray d'hypersurfaces nodales aléatoires sur le tore $\mathbb{T}^n$. Les auteurs y calculent l'espérance de cette quantité, à k fixé, et l'asymptotique de sa variance lorsque $k \to +\infty$, pour $n = 2$ et $n \geqslant 5$. La mesure de Leray de $Z_f \subset \mathbb{T}^n$ est définie comme :

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \operatorname{Vol}\left(\{x \in \mathbb{T}^n \mid |f(x)| < \varepsilon\}\right). \tag{1.2.5}$$

Dans [RW08], Rudnick et Wigman calculent l'espérance du volume d'une hypersurface nodale dans $\mathbb{T}^n$ ($n \geqslant 2$) et donnent une borne asymptotique sur la variance de ce volume. Ce résultat est ensuite précisé dans [KKW13], où les auteurs déterminent l'asymptotique exacte pour la variance du volume d'une hypersurface en dimension ambiante $n = 2$. Enfin, [MPRW16] décrit la convergence en loi de la longueur d'une courbe nodale dans $\mathbb{T}^2$. Dans les deux derniers papiers cités, il n'y a pas de convergence lorsque $k \to +\infty$, mais seulement pour des sous-suites de valeurs propres tendant vers l'infini.

La longueur des courbes nodales associées à une onde aléatoire monochromatique sur la sphère euclidienne $\mathbb{S}^2$ a été étudié par Wigman dans [Wig10, Wig12]. Soit f_k un vecteur gaussien standard dans $\ker(\Delta - \lambda_k \operatorname{Id})$ où $\lambda_k = k(k+1)$ est la k-ième valeur propre de Δ sur $\mathbb{S}^2$, on note $|\mathrm{d}V_k|$ la mesure d'intégration sur le lieu des zéros de f_k. Wigman obtient alors le résultat suivant.

Théorème 1.2.7 (Wigman, 2010)**.** *Soit $\phi \in L^\infty(\mathbb{S}^2)$ une fonction-test à variations bornées, on a :*

$$\operatorname{Var}(\langle |\mathrm{d}V_k|\,, \phi\rangle) = \frac{1}{128\pi}\left(\int_{\mathbb{S}^2} \phi^2\right) \ln k + O(1),$$

lorsque k tend vers l'infini.

En particulier, la variance du volume de l'hypersurface nodale définie par f_k est équivalente à $\frac{1}{32}\ln k$ lorsque k tend vers l'infini. Précédemment, dans [Wig09], Wigman avait obtenu l'asymptotique de la variance de la mesure de Leray (voir (1.2.5)) d'une hypersurface nodale définie par une onde aléatoire monochromatique sur $\mathbb{S}^n$, ainsi qu'une borne sur la vitesse de croissance de la variance du volume.

En lien avec le théorème 1.2.6, Gayet et Welschinger calculent dans [GW14a] une majoration asymptotique d'ordre $\lambda^{\frac{n}{2}}$, quand $\lambda \to +\infty$, pour les nombres de Betti moyens de l'hypersurface Z_f, dans le cas des ondes riemanniennes aléatoires avec $r = 1$. Leur résultat est en fait valable pour les hypersurfaces nodales définies comme lieu des zéros d'une combinaison linéaire aléatoire de fonctions propres d'un opérateur pseudo-différentiel elliptique. Leur preuve fait en particulier apparaître le résultat suivant. Soit p une fonction de Morse sur la variété ambiante M, alors presque surement $p_{/Z_f}$ est encore de Morse. On note alors $N_i(Z_f)$ le nombre de points critiques d'indice $i \in \{0, \dots, n-1\}$ de $p_{/Z_f}$.

Théorème 1.2.8 (Gayet–Welschinger, 2014)**.** *Pour tout $i \in \{0, \dots, n-1\}$ on a l'asymptotique suivante lorsque $\lambda \to +\infty$:*

$$\mathbb{E}[N_i(Z_f)] \sim \lambda^{\frac{n}{2}} C_n(i) \operatorname{Vol}(M),$$

où $C_n(i) > 0$ est une constante ne dépendant que de i et n.

Nous renvoyons à [GW14a, cor. 0.3] pour la définition précise de la constante $C_n(i)$. Par la théorie de Morse, ce théorème montre que :

$$\mathbb{E}[\chi(Z_f)] = \sum_{i=0}^{n-1} (-1)^i \mathbb{E}[N_i(Z_f)] = \left(\sum_{i=0}^{n-1} (-1)^i C_n(i) \right) \lambda^{\frac{n}{2}} \operatorname{Vol}(M) + o(\lambda^{\frac{n}{2}}).$$

Il a été observé, après la publication de [Let16a], qu'il est possible de calculer directement la valeur de $\left(\sum_{i=0}^{n-1} (-1)^i C_n(i) \right)$ à partir de la définition des $C_n(i)$. Cela permet d'établir le théorème 1.2.6 pour $r = 1$. La preuve du théorème 1.2.6 que nous proposons dans le chapitre 2 est indépendante du théorème 1.2.8.

Citons aussi le résultat de Dang et Rivière [DR15]. Dans le cadre de ondes riemanniennes aléatoires, et sous les hypothèses supplémentaires que M soit connexe et orientée, ils calculent l'asymptotique de la moyenne du cycle conormal de Z_f, en codimension $r = 1$. Ce résultat leur permet de donner une nouvelle démonstration du théorème 1.2.6, sous ces hypothèses. Le cycle conormal de l'hypersurface Z_f est défini comme :

$$N^*(Z_f) = \{(x, \xi) \in T^*M \mid f(x) = 0 \text{ et il existe } t \neq 0 \text{ tel que } \xi = t d_x f\}.$$

Ce cycle conormal définit presque surement un courant de dimension n sur T^*M (voir [DR15] pour les subtilités liées au choix d'une orientation sur $N^*(Z_f)$).

Théorème 1.2.9 (Dang–Rivière, 2015)**.** *Soit (M, g) une variété riemannienne de dimension n connexe et orientée, soit $f \sim \mathcal{N}(0, \mathrm{Id})$ dans V_λ, on a l'asymptotique suivante quand $\lambda \to +\infty$:*

$$\mathbb{E}[N^*(Z_f)] = C_n \left(\frac{\lambda}{n+2} \right)^{\frac{n}{2}} \pi^\star \Omega_M + O(\lambda^{\frac{n-1}{2}}),$$

où $C_n = \dfrac{2(-1)^{\frac{n-1}{2}}}{\pi \operatorname{Vol}(\mathbb{S}^n)}$ si n est impair et $C_n = 0$ sinon. Ici, $\pi : T^*M \to M$ est la projection canonique et Ω_M est la forme volume induite par la métrique riemannienne sur M. L'égalité a lieu en tant que courants de dimension n sur T^*M.

Dans leur livre [TA07], Taylor et Adler s'intéressent à la caractéristique d'Euler de surniveaux (excursion sets) d'un champ gaussien sur une variété. Pour énoncer leur résultat il nous faut évoquer rapidement les courbures de Lipschitz–Killing. Nous ne donnons pas la définition précise de ces quantités ici. On peut la trouver dans [TA07], formule (12.3.4). Soit (M, g) une variété riemannienne lisse fermée de dimension n, ses courbures de Lipschitz–Killing sont $n+1$ réels, notés $\mathcal{L}_0(M, g), \dots, \mathcal{L}_n(M, g)$, qui sont définis comme des intégrales sur M de certaines quantités dépendant de la courbure de Riemann de (M, g). Deux de ces quantités sont très simples à interpréter : $\mathcal{L}_n(M, g)$ est le volume riemannien de M et $\mathcal{L}_0(M, g) = \chi(M)$. Cette seconde relation découle directement de la définition de $\mathcal{L}_0(M, g)$ et du théorème de Chern–Gauss–Bonnet (voir thm. 2.4.3).

Théorème 1.2.10 (Taylor–Adler, 2003). *Soient M une variété lisse fermée et $(f(x))_{x\in M}$ un champ gaussien lisse centré tel que pour tout $x \in M$, $\mathrm{Var}(f(x)) = 1$. Supposons de plus que, pour tout $x \in M$, la distribution de $d_x f$ ne soit pas dégénérée. Alors, pour tout $u \in \mathbb{R}$, on a :*

$$\mathbb{E}[\chi(\{x \in M \mid f(x) \geqslant u\})] = \sum_{i=0}^{n} \mathcal{L}_i(M, g_f)\rho_i(u),$$

où g_f est la métrique induite par le champ f sur M et $\rho_0, \dots, \rho_n$ sont $n+1$ fonctions de $\mathbb{R}$ dans $\mathbb{R}$, indépendantes de M et f.

Ce résultat est le théorème 12.4.1 de [TA07]. Les fonctions $\rho_0, \dots, \rho_n$ sont explicites et leurs expressions sont données par [TA07, formule (12.4.2)]. La métrique g_f qui apparaît dans ce théorème est définie par :

$$(g_f)_x(u, v) = \mathbb{E}[(d_x f \cdot u)(d_x f \cdot v)],$$

pour tout $x \in M$ et tout $u, v \in T_x M$. La condition de non dégénérescence sur la distribution de $d_x f$ assure que g_f est bien une métrique riemannienne.

Ce résultat s'applique en particulier pour calculer la caractéristique d'Euler moyenne de $f^{-1}([0, +\infty[)$ avec $f \sim \mathcal{N}(0, \mathrm{Id})$ dans V_λ. Il faut pour cela remplacer f par

$$x \longmapsto \frac{f(x)}{\sqrt{\mathrm{Var}(f(x))}},$$

ce qui ne modifie pas $f^{-1}([0, +\infty[)$. Ensuite, si la dimension ambiante n est impaire, on a $\chi(Z_f) = 2\chi\left(f^{-1}([0, +\infty[)\right)$. On déduit alors du théorème 1.2.10 que :

$$\mathbb{E}[\chi(Z_f)] = 2\sum_{i=0}^{n} \mathcal{L}_i(M, g_f)\rho_i(0).$$

Cela fournit une expression de la caractéristique d'Euler moyenne d'une hypersurface nodale dans le modèle des ondes riemanniennes aléatoires, en fonction de quantités dépendant fortement de la métrique g_f (donc de λ dans le cas qui nous intéresse). Nous pensons qu'il est possible de redémontrer le théorème 1.2.6 à partir de la relation ci-dessus. Pour cela il serait nécessaire d'estimer les quantités $\mathcal{L}_i(M, g_f)$ en fonction de λ lorsque $\lambda \to +\infty$.

Dans les prépublications [CMW15, CM16], les auteurs étudient la distribution de la caractéristique d'Euler de surniveaux d'une onde aléatoire monochromatique sur $\mathbb{S}^2$ par ce genre de méthodes.

Jusqu'à présent, les résultats que nous avons évoqués concernent des quantités additives (volume, caractéristique d'Euler, nombre de points critiques d'une fonction de Morse) ou "locales" pour reprendre les termes de [Ana16]. Une autre quantité additive très étudiée dans ce domaine est le nombre de points critiques de la fonction aléatoire f, voir [CMW14, CW15] et les travaux de Nicolaescu [Nic15a, Nic15b, Nic15c, Nic16].

Les quantités additives ont l'avantage que le calcul de leur espérance est, en principe, accessible par une formule de type Kac–Rice (voir thm. 2.5.3 et thm. 3.4.4). Ceci sera détaillé dans la section 1.4. En revanche, les quantités "globales" telles que le nombre de composantes connexes, ou plus généralement les nombres de Betti, ne sont pas accessibles par ce genre de méthodes. Dans la fin de cette section, nous présentons quelques résultats frappants impliquant des quantités topologiques globales au sens précédent. Les travaux concernant la topologie des sous-variétés nodales aléatoires ont récemment fait l'objet d'un exposé au séminaire Bourbaki par Nalini Anantharaman. Nous renvoyons au texte [Ana16] accompagnant cet exposé pour une revue plus détaillée de ces résultats.

Considérons le modèle des ondes aléatoires monochromatiques sur $\mathbb{S}^2$ munie de sa métrique euclidienne. Rappelons que dans ce cadre $\lambda_k = k(k+1)$ est la k-ième valeur propre du laplacien. Pour $f \in \ker(\Delta - \lambda_k \operatorname{Id})$ on note $b_0(Z_f)$ le nombre de composantes connexes de Z_f. Dans [NS09], Nazarov et Sodin démontrent que $b_0(Z_f)$ se concentre en probabilité autour de sa moyenne.

Théorème 1.2.11 (Nazarov–Sodin, 2009)**.** *Il existe $a > 0$ tel que :*

$$\frac{1}{k^2}\mathbb{E}[b_0(Z_f)] \xrightarrow[k\to+\infty]{} a.$$

De plus, pour tout $\varepsilon > 0$ il existe $c(\varepsilon)$ et $C(\varepsilon)$ strictement positifs tels que :

$$\mathbb{P}\left(\left|\frac{1}{k^2}b_0(Z_f) - a\right| > \varepsilon\right) \leqslant C(\varepsilon)e^{-kc(\varepsilon)}.$$

Dans [NS15], les auteurs obtiennent un résultat permettant de démontrer la convergence L^1 du nombre de composantes connexes dans un cadre très général. Ce résultat s'applique en particulier au modèle de bande passante décrit au début de cette section et on peut en déduire le théorème suivant (voir [Ana16, thm. 0.11]).

Théorème 1.2.12 (Nazarov–Sodin, 2015)**.** *Soit (M, g) une variété riemannienne fermée de dimension n, soit $\ell \in [0, 1[$ et soit f un vecteur gaussien standard dans V_λ^ℓ. Alors il existe une constante $a_{n,\ell}$ ne dépendant que de n et ℓ telle que :*

$$\mathbb{P}\left(\left|\frac{1}{\lambda^{\frac{n}{2}}}b_0(Z_f) - a_{n,\ell}\right| > \varepsilon\right) \xrightarrow[\lambda\to+\infty]{} 0.$$

Les idées de [NS15] ont aussi été exploités par Rozenshein [Roz16] pour étudier le nombre de composantes connexes d'hypersurfaces nodales dans le cas des ondes aléatoires arithmétiques. De même, Beliaev et Wigman [BW16] utilisent des idées proches pour étudier les composantes connexes d'hypersurfaces nodales, dans le modèle de bande passante sur une variété riemannienne et dans le cas des ondes aléatoires monochromatiques sur la sphère euclidienne.

Nous avons déjà mentionné que Gayet et Welschinger prouvent une majoration asymptotique pour les nombres de Betti d'une hypersurface nodale dans le cas des ondes riemanniennes aléatoires (voir [GW14a]). Dans [GW15b] ils démontrent une borne inférieure asymptotique, du même ordre de grandeur $\lambda^{\frac{n}{2}}$, pour ces nombres de Betti. Leur preuve repose sur le résultat local suivant.

Théorème 1.2.13 (Gayet–Welschinger, 2015)**.** *Soit (M, g) une variété riemannienne fermée de dimension n et soit Σ une hypersurface fermée de $\mathbb{R}^n$, pas nécessairement connexe. Soit $f \sim \mathcal{N}(0, \operatorname{Id})$ dans V_λ. Pour tout $x \in M$, on note $B_\lambda(x, R)$ la boule géodésique ouverte de centre x et de rayon $\frac{R}{\sqrt{\lambda}}$. Alors il existe une constante $c_\Sigma(R) \geqslant 0$ ne dépendant que de Σ et R telle que, pour tout $x \in M$, la probabilité que $Z_f \cap B_\lambda(x, R)$ contienne une hypersurface difféomorphe à Σ est minorée par $c_\Sigma(R)$.*

De plus $c_\Sigma(R)$ est strictement positive pour R assez grand.

Remarque 1.2.14. Gayet et Welschinger prouvent en fait une généralisation de ce théorème aux hypersurfaces nodales définies comme lieu d'annulation d'une combinaison linéaire aléatoire de fonctions propres d'un opérateur pseudo-différentiel elliptique.

Citons enfin que Sarnak et Wigman [SW15] ont étudié la fréquence d'apparition des différents types de difféomorphismes d'hypersurfaces dans Z_f, ainsi que la façon dont les différentes composantes connexes de Z_f s'agencent (voir aussi [CS14]). Ils ont mené cette étude dans le cas d'hypersurfaces nodales aléatoires sur la sphère $\mathbb{S}^n$ munie d'une métrique riemannienne lisse quelconque, dans le modèle de bande passante décrit ci-dessus. L'énoncé précis de leur théorème demande l'introduction d'un formalisme assez important et nous renvoyons à [SW15] ou [Ana16, thm 0.14]. Initialement ce résultat avait été annoncé dans [SW13] pour une variété riemannienne connexe fermée quelconque.

1.3 Cadre algébrique réel

Dans cette section nous commençons par décrire le modèle de sous-variétés algébriques réelles avec lequel nous travaillons (voir 1.3.1). Le section 1.3.2 décrit un cas particulier simple de notre modèle. Nos résultats dans ce cadre algébrique sont énoncés dans la section 1.3.3. Enfin nous présentons un certain nombre de résultats en rapport avec les nôtres dans la section 1.3.4.

1.3.1 Sous-variétés algébriques réelles aléatoires

Le cadre algébrique que nous considérons est d'abord un cadre algébrique complexe, auquel on ajoute une structure réelle. Nous commençons donc naturellement par décrire un modèle de sous-variétés algébriques complexes. Ce modèle a fait l'objet de nombreux travaux de Shiffman et Zelditch qui seront décris dans la section 1.3.4. Nous renvoyons à [GH94] pour tous les résultats classiques de géométrie complexe que nous utilisons.

Soit $\mathcal{X}$ une variété complexe projective de dimension complexe $n \geqslant 1$. Soit $\mathcal{L} \to \mathcal{X}$ un fibré holomorphe en droites complexes. On suppose que $\mathcal{L}$ est équipé d'une métrique hermitienne $h_{\mathcal{L}}$ à courbure positive, en particulier $\mathcal{L}$ est ample. Rappelons que la *courbure* de $(\mathcal{L}, h_{\mathcal{L}})$ est une $(1,1)$-forme $\Theta_{\mathcal{L}}$ que l'on peut décrire localement comme suit. Si ζ est une section holomorphe locale de $\mathcal{L}$ définie sur un ouvert Ω où elle ne s'annule pas, alors :

$$(\Theta_{\mathcal{L}})_{/\Omega} = -\partial\overline{\partial} \log\left(h_{\mathcal{L}}(\zeta, \zeta)\right).$$

Cette définition ne dépend pas du référentiel holomorphe local ζ choisi. On dit que $\Theta_{\mathcal{L}}$ est *positive* si, pour tout v non nul dans l'espace tangent holomorphe $T_x\mathcal{X}'$, $\Theta_{\mathcal{L}}(v, \bar{v}) > 0$. De façon équivalente, la $(1,1)$-forme réelle $\omega = \frac{i}{2}\Theta_{\mathcal{L}}$ est une forme de Kähler. Par abus de langage nous dirons que ω est la courbure de $\mathcal{L}$.

En notant J la structure complexe sur $\mathcal{X}$, la forme de Kähler ω définit une métrique riemannienne g sur $\mathcal{X}$ par $g = \omega(\cdot, J\cdot)$. Elle induit également une métrique hermitienne $g_{\mathbb{C}} = g - i\omega$ et une forme volume $\mathrm{d}V_{\mathcal{X}} = \frac{\omega^n}{n!}$ sur $\mathcal{X}$. Notons que la mesure de volume définie par $\mathrm{d}V_{\mathcal{X}}$ coïncide avec la mesure riemannienne définie par g.

Remarque 1.3.1. Certains auteurs définissent ω comme $\frac{i}{2\pi}\Theta_{\mathcal{L}}$, de sorte que ω représente la première classe de Chern de $\mathcal{L}$. Avec notre normalisation, dans le cas des polynômes de Kostan–Shub–Smale décrit dans la section 1.3.2, la métrique induite par ω sur $\mathbb{CP}^n$ sera la métrique quotient de la métrique euclidienne sur la sphère.

Soient $r \in \{1, \dots, n\}$ et $\mathcal{E} \to \mathcal{X}$ un fibré holomorphe de rang r, muni d'une métrique hermitienne $h_{\mathcal{E}}$. Pour tout $d \in \mathbb{N}$, on définit une métrique hermitienne $h_d = h_{\mathcal{E}} \otimes (h_{\mathcal{L}})^{\otimes d}$ sur $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. La métrique h_d et la forme de Kähler ω permettent alors de définir un produit scalaire L^2 hermitien sur l'espace $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ des sections lisses de $\mathcal{E} \otimes \mathcal{L}^d$. Précisément, on définit pour tout s_1 et $s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d)$:

$$\langle s_1 \,, s_2 \rangle = \int_{x \in \mathcal{X}} h_d(s_1(x), s_2(x))\, \mathrm{d}V_{\mathcal{X}}. \tag{1.3.1}$$

Par ailleurs on note $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ l'espace des sections holomorphes globales de $\mathcal{E} \otimes \mathcal{L}^d$. On sait, par exemple par la théorie de Hodge, que $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est de dimension finie (voir [MM07, thm. 1.4.1]). Notons N_d la dimension de cet espace, les estimations diagonales sur le noyau de Bergman (voir section 1.4) permettent de montrer que, lorsque d tend vers l'infini,

$$N_d \sim \left(\frac{d}{\pi}\right)^n r \operatorname{Vol}(\mathcal{X}). \tag{1.3.2}$$

Définition 1.3.2. Dans la suite, nous appelerons *vecteur gaussien complexe standard* dans $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ une section aléatoire $s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ suivant une loi $\mathcal{N}(0, \mathrm{Id})$ dans $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ en tant qu'espace vectoriel sur $\mathbb{R}$, muni du produit scalaire euclidien induit par (1.3.1). En d'autres termes, si $(s_1, \ldots, s_{N_d})$ est une base orthonormée de $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, alors

$$s = \sum_{j=1}^{N_d} (a_j + ib_j) s_j,$$

où les a_j et b_j sont des variables aléatoires réelles indépendantes de loi $\mathcal{N}(0, 1)$. On note $s \sim \mathcal{N}_{\mathbb{C}}(0, \mathrm{Id})$.

Soit $s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien complexe standard, on note $Z_d^{\mathbb{C}}$ le lieu d'annulation de s dans $\mathcal{X}$. Nous réservons la notation Z_d au lieu d'annulation réel définit plus loin. Presque surement, $Z_d^{\mathbb{C}}$ est une sous-variété complexe de codimension r de $\mathcal{X}$ donc définit un courant de bi-degré (r, r).

Dans cette situation complexe, un certain nombre de quantités liées à $Z_d^{\mathbb{C}}$ sont en fait déterministes. Par exemple, dans le cas où $r = 1$ et $\mathcal{E}$ est trivial, on s'intéresse à une section holomorphe globale s de $\mathcal{L}^d \to \mathcal{X}$. Alors s s'annule le long d'un diviseur qui définit une classe d'homologie Poincaré-duale de $dc_1(\mathcal{L})$, la première classe de Chern de $\mathcal{L}^d$ (voir [GH94, p. 141] par exemple). De plus $c_1(\mathcal{L})$ est représentée par $\frac{1}{\pi}\omega$. Dans le cas où $s^{-1}(0)$ est une sous-variété lisse on a donc :

$$\begin{aligned}
\operatorname{Vol}\left(s^{-1}(0)\right) &= \int_{s^{-1}(0)} \frac{\omega^{n-1}}{(n-1)!} \\
&= \int_{\mathcal{X}} \frac{d}{\pi}\omega \wedge \frac{\omega^{n-1}}{(n-1)!} \\
&= \frac{nd}{\pi} \operatorname{Vol}(\mathcal{X}).
\end{aligned}$$

On a donc $\operatorname{Vol}\left(s^{-1}(0)\right) = \frac{nd}{\pi} \operatorname{Vol}(\mathcal{X})$ presque surement, ce qui est à rapprocher du fait qu'un polynôme homogène de degré d en deux variables a presque surement d racines distinctes dans $\mathbb{CP}^1$.

Nous n'avons donc pas besoin d'introduire d'aléa pour comprendre le comportement de quantités comme le volume de $s^{-1}(0)$. Les résultats que nous présenterons dans ce cadre (voir section 1.3.4) portent plutôt sur des phénomènes d'équidistribution en moyenne de $Z_d^{\mathbb{C}}$ dans $\mathcal{X}$.

Passons maintenant à la description de notre cadre de travail algébrique réel. C'est aussi le cadre des articles de Gayet et Welschinger [GW11a, GW14b, GW14c, GW15a, GW16]. L'idée de base est de rajouter des structures réelles compatibles au cadre algébrique complexe dont nous venons de discuter. Nous donnons un exemple concret de ce cadre dans la prochaine section 1.3.2.

Soit $c_{\mathcal{X}}$ une structure réelle sur $\mathcal{X}$, c'est-à-dire une involution anti-holomorphe de $\mathcal{X}$. Le *lieu réel* de $\mathcal{X}$ est l'ensemble des points fixes de $c_{\mathcal{X}}$. Nous supposons dorénavant que ce lieu réel est non vide et nous le notons M (dans le chapitre 2 il est noté $\mathbb{R}\mathcal{X}$). Alors, M est une sous-variété lisse fermée de dimension réelle n de $\mathcal{X}$ (voir [Sil89, chap. 1])

Soit $c_{\mathcal{L}}$ une structure réelle sur $\mathcal{L}$ compatible avec $c_{\mathcal{X}}$. C'est-à-dire, en notant $\pi_{\mathcal{L}}$ la projection de $\mathcal{L} \to \mathcal{X}$, $c_{\mathcal{L}}$ est une involution anti-holomorphe de $\mathcal{L}$ telle que $\pi_{\mathcal{L}} \circ c_{\mathcal{L}} = c_{\mathcal{X}} \circ \pi_{\mathcal{L}}$ et qui est $\mathbb{C}$-anti-linéaire dans les fibres. De même, on suppose que $\mathcal{E}$ est équipé d'une structure $c_{\mathcal{E}}$ compatible avec $c_{\mathcal{X}}$. Alors, pour tout $d \in \mathbb{N}$, $c_d = c_{\mathcal{X}} \otimes (c_{\mathcal{L}})^{\otimes d}$ définit une structure réelle sur $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, compatible avec $c_{\mathcal{X}}$.

Définition 1.3.3. On dit qu'une section s de $\mathcal{E} \otimes \mathcal{L}^d$ est *réelle* si elle est équivariante pour les structures réelles, c'est-à-dire si $c_d \circ s \circ c_{\mathcal{X}} = s$. On note $\mathbb{R}\Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ l'ensemble des sections réelles de $\mathcal{E} \otimes \mathcal{L}^d$. On note également

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d) \cap H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d).$$

Lemme 1.3.4. *Pour tout $d \in \mathbb{N}$, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est un sous-espace réel de $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ de dimension réelle N_d.*

Démonstration. Soit $C : s \mapsto c_d \circ s \circ c_{\mathcal{X}}$. Tout d'abord remarquons que $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est stable par C, car c_d et $c_{\mathcal{X}}$ sont toutes deux anti-holomorphes. Alors la restriction de C à $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est une involution $\mathbb{C}$-anti-linéaire. Comme c'est une involution, on a

$$H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \ker(C - \mathrm{Id}) \oplus \ker(C + \mathrm{Id}),$$

sur $\mathbb{R}$. Par ailleurs, comme C est $\mathbb{C}$-anti-linéaire, la multiplication par i échange $\ker(C - \mathrm{Id})$ et $\ker(C + \mathrm{Id})$. Ainsi $\ker(C - \mathrm{Id})$ et $\ker(C + \mathrm{Id})$ ont même dimension réelle, égale à N_d. Comme $\ker(C - \mathrm{Id}) = \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, le lemme est prouvé. □

Il nous faut maintenant imposer des conditions de compatibilité entre les structures réelles et les métriques hermitiennes sur $\mathcal{L}$ et $\mathcal{E}$. Nous supposons désormais que $h_{\mathcal{L}}$ et $h_{\mathcal{E}}$ sont telles que :

$$c_{\mathcal{L}}^{\star}(h_{\mathcal{L}}) = \overline{h_{\mathcal{L}}} \qquad \text{et} \qquad c_{\mathcal{E}}^{\star}(h_{\mathcal{E}}) = \overline{h_{\mathcal{E}}}.$$

Ces conditions miment la situation modèle d'un fibré trivial au-dessus d'un ouvert de $\mathbb{C}^n$, équipé de la conjugaison usuelle et du produit scalaire hermitien usuel dans chaque fibre. Une conséquence immédiate est que, pour tout $d \in \mathbb{N}$, h_d est compatible avec c_d. Cela a aussi des conséquences sur la métrique de Kähler. Nous aurons besoin des relations suivantes dans le chapitre 3.

Lemme 1.3.5. *Si $h_{\mathcal{L}}$ est compatible avec les structures réelles $c_{\mathcal{X}}$ et $c_{\mathcal{L}}$, on a :*

$$c_{\mathcal{X}}^{\star}(\omega) = -\omega, \qquad c_{\mathcal{X}}^{\star}(g) = g \qquad \textit{et} \qquad c_{\mathcal{X}}^{\star}(g_{\mathbb{C}}) = \overline{g_{\mathbb{C}}}.$$

En particulier, $c_{\mathcal{X}}$ est une isométrie de $(\mathcal{X}, g)$.

Démonstration. Soit $x \in \mathcal{X}$ et ζ une section holomorphe locale de $\mathcal{L}$, définie sur un voisinage Ω de x. Si $x \notin M$, on peut supposer que $c_{\mathcal{X}}(\Omega) \cap \Omega = \emptyset$. On peut alors étendre ζ à $c_{\mathcal{X}}(\Omega) \cup \Omega$ en posant $\zeta(c_{\mathcal{X}}(x)) = c_{\mathcal{L}}(\zeta(x))$ pour tout $x \in \Omega$. On obtient ainsi une section holomorphe de $\mathcal{L}$ au-dessus de $c_{\mathcal{X}}(\Omega) \cup \Omega$, ne s'annulant pas sur ce domaine, et y vérifiant $c_{\mathcal{L}} \circ \zeta = \zeta \circ c_{\mathcal{X}}$.

Si $x \in M$, alors on peut supposer que $c_{\mathcal{X}}(\Omega) = \Omega$, quitte à restreindre Ω. Ensuite, on peut supposer que ζ est telle que $c_{\mathcal{L}} \circ \zeta = \zeta \circ c_{\mathcal{X}}$ sur Ω. Si $c_{\mathcal{L}}(\zeta(x)) + \zeta(x) \neq 0$, il suffit de remplacer ζ par $\zeta + c_{\mathcal{L}} \circ \zeta \circ c_{\mathcal{X}}$, qui ne s'annule pas en x, et de restreindre éventuellement Ω, tout en s'assurant qu'on conserve un voisinage stable par $c_{\mathcal{X}}$. Si $c_{\mathcal{L}}(\zeta(x)) + \zeta(x) = 0$, alors $c_{\mathcal{L}}(i\zeta(x)) + i\zeta(x) = 2i\zeta(x) \neq 0$. On peut alors remplacer ζ par $i\zeta + c_{\mathcal{L}} \circ (i\zeta) \circ c_{\mathcal{X}}$ qui convient. Dans tous les cas, il existe un voisinage de Ω de x stable par $c_{\mathcal{X}}$ et une section ζ holomorphe de $\mathcal{L}$ au-dessus de Ω, ne s'annulant pas, et telle que $c_{\mathcal{L}} \circ \zeta = \zeta \circ c_{\mathcal{X}}$.

Sur Ω, on a alors :

$$\begin{aligned} c_{\mathcal{X}}^{\star}\left(h_{\mathcal{L}}(\zeta,\zeta)\right) &= h_{\mathcal{L}}(\zeta \circ c_{\mathcal{X}}, \zeta \circ c_{\mathcal{X}}) \\ &= h_{\mathcal{L}}(c_{\mathcal{L}} \circ \zeta, c_{\mathcal{L}} \circ \zeta) \\ &= c_{\mathcal{L}}^{\star}\left(h_{\mathcal{L}}\right)(\zeta,\zeta) \\ &= \overline{h_{\mathcal{L}}(\zeta,\zeta)} \\ &= h_{\mathcal{L}}(\zeta,\zeta). \end{aligned}$$

Puis, comme $c_{\mathcal{X}}$ est anti-holomorphe,

$$\begin{aligned} c_{\mathcal{X}}^{\star}\omega &= \frac{i}{2} c_{\mathcal{X}}^{\star} \partial\overline{\partial} \log\left(h_{\mathcal{L}}(\zeta,\zeta)\right) \\ &= \frac{i}{2} \overline{\partial}\partial \left(c_{\mathcal{X}}^{\star} \log\left(h_{\mathcal{L}}(\zeta,\zeta)\right)\right) \\ &= -\frac{i}{2} \partial\overline{\partial} \log\left(h_{\mathcal{L}}(\zeta,\zeta)\right) \\ &= -\omega. \end{aligned}$$

Maintenant que la première relation est établie, on a :

$$\begin{aligned} (c_{\mathcal{X}}^{\star} g)_x &= c_{\mathcal{X}}^{\star}(\omega(\cdot, J\cdot))_x \\ &= \omega_{c_{\mathcal{X}}(x)}\left(d_x c_{\mathcal{X}} \cdot, J \circ d_x c_{\mathcal{X}} \cdot\right) \\ &= \omega_{c_{\mathcal{X}}(x)}\left(d_x c_{\mathcal{X}} \cdot, -d_x c_{\mathcal{X}} \circ J\cdot\right) \\ &= -\left(c_{\mathcal{X}}^{\star}\omega\right)_x (\cdot, J\cdot) \\ &= \omega_x(\cdot, J\cdot) \\ &= g_x. \end{aligned}$$

Enfin, comme $g_{\mathbb{C}} = g - i\omega$, la dernière relation découle des deux premières. □

Lemme 1.3.6. *Si les métriques $h_{\mathcal{L}}$ et $h_{\mathcal{E}}$ sont compatibles avec les structures réelles, alors la restriction du produit scalaire hermitien* (1.3.1) *à l'espace vectoriel réel $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est un produit scalaire euclidien pour tout $d \in \mathbb{N}$.*

Démonstration. Il s'agit de vérifier que, pour tout s_1 et $s_2 \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, $\langle s_1 \, , s_2 \rangle$ est réel. Comme $h_{\mathcal{L}}$ et $h_{\mathcal{E}}$ sont compatibles avec les structures réelles alors h_d l'est aussi. Soit s_1 et $s_2 \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, on a alors :

$$\begin{aligned} \overline{\langle s_1 \, , s_2 \rangle} &= \int_{x \in \mathcal{X}} \overline{h_d(s_1(x), s_2(x))} \, \mathrm{d}V_{\mathcal{X}} \\ &= \int_{x \in \mathcal{X}} c_d^{\star}(h_d)(s_1(x), s_2(x)) \, \mathrm{d}V_{\mathcal{X}} \\ &= \int_{x \in \mathcal{X}} h_d(c_d \circ s_1(x), c_d \circ s_2(x)) \, \mathrm{d}V_{\mathcal{X}} \\ &= \int_{x \in \mathcal{X}} h_d(s_1(c_{\mathcal{X}}(x)), s_2(c_{\mathcal{X}}(x))) \, \mathrm{d}V_{\mathcal{X}} \\ &= \int_{x \in \mathcal{X}} h_d(s_1(x), s_2(x)) \, \mathrm{d}V_{\mathcal{X}} \\ &= \langle s_1 \, , s_2 \rangle \, , \end{aligned}$$

où on obtient la dernière ligne par un changement de variable, en utilisant le fait que $c_{\mathcal{X}}$ est une isométrie (voir lemme 1.3.5), donc ne modifie pas la mesure riemannienne $\mathrm{d}V_{\mathcal{X}}$. $\square$

Ainsi, pour tout $d \in \mathbb{N}$, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est un sous-espace vectoriel de dimension N_d de $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ et (1.3.1) définit un produit scalaire euclidien sur $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Nous démontrons dans la section 2.3.3 (voir corollaire 2.3.10) que $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ est 0-ample pour tout d assez grand. Nous retrouvons donc le cadre général décrit dans la section 1.1.

Soit $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ une section aléatoire de loi gaussienne standard, on note Z_d le lieu des zéros réels de s : $Z_d = s^{-1}(0) \cap M$. Alors, pour tout d assez grand, Z_d est presque surement une sous-variété lisse de M de codimension r (voir section 2.2.6) et on note $|\mathrm{d}V_d|$ la mesure riemannienne d'intégration sur Z_d. Nous obtenons alors des résultats asymptotiques lorsque $d \to +\infty$, portant sur la caractéristique d'Euler moyenne de Z_f, ainsi que sur la moyenne et la variance de $\mathrm{Vol}(Z_f)$. Ces résultats sont énoncés dans la section 1.3.3.

Remarque 1.3.7. Même lorsque l'on considère une section réelle, le domaine d'intégration dans la définition du produit scalaire (1.3.1) est la variété complexe $\mathcal{X}$ entière. Ce modèle est parfois appelé modèle de *Fubini–Study complexe*, par opposition au modèle de *Fubini–Study réel* où l'intégrale définissant le produit scalaire (1.3.1) porte cette fois seulement sur le lieu réel M. Ce dernier modèle a été étudié, entre autre, par Lerario et Lundberg [LL15] dans le cas où $\mathcal{X} = \mathbb{CP}^n$.

1.3.2 Les polynômes de Kostlan–Shub–Smale

Dans cette section nous présentons un exemple du cadre algébrique réel que nous venons de décrire. En particulier, cela montre qu'il existe des exemples non triviaux de ce modèle, ce qui n'est pas totalement évident vu les conditions de compatibilité que l'on impose entre les différentes structures. Précisons que tout ce qui suit est classique, voir [Kos93, Kos02, SZ99].

On se place dans le cas où $\mathcal{X} = \mathbb{CP}^n$, muni de la structure réelle héritée de la conjugaison dans $\mathbb{C}^{n+1}$. Le lieu réel de $\mathbb{CP}^n$ est, sans surprise $\mathbb{RP}^n$.

Sur $\mathbb{CP}^n$, on dispose d'un fibré holomorphe en droites naturel, le fibré *tautologique* :

$$\mathcal{O}(-1) = \left\{(x, D) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \;\middle|\; x \in D\right\}.$$

La conjugaison usuelle dans $\mathbb{C}^{n+1}$ induit alors sur $\mathcal{O}(-1)$ une structure réelle qui est tautologiquement compatible avec celle de $\mathbb{CP}^n$. Ensuite, le produit scalaire hermitien usuel de $\mathbb{C}^{n+1}$ définit une métrique hermitienne sur le fibré trivial $\mathbb{C}^{n+1} \times \mathbb{CP}^n \to \mathbb{CP}^n$ qui se restreint en une métrique hermitienne sur $\mathcal{O}(-1)$. Comme le produit scalaire de $\mathbb{C}^{n+1}$ est compatible avec la conjugaison (c'est de cela qu'on s'est inspiré pour définir la notion de compatibilité), la métrique et la structure réelle de $\mathcal{O}(-1)$ sont compatibles.

Ce fibré n'est pas positif, mais son fibré dual $\mathcal{O}(1)$ l'est. On choisit alors $\mathcal{L} = \mathcal{O}(1)$, qui est souvent appelé *fibré des hyperplans*. Il hérite d'une structure réelle naturelle et d'une métrique hermitienne $h_{\mathcal{L}}$ duale de la métrique tautologique sur $\mathcal{O}(-1)$. La forme de Kähler associée à $h_{\mathcal{L}}$ est la forme de Fubini–Study ω_{FS} (voir [GH94, p. 31]) normalisée de sorte que la métrique riemannienne associée soit la métrique quotient de la métrique euclidienne sur $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$.

Pour tout $d \in \mathbb{N}^*$, $\mathcal{L}^d$ est noté $\mathcal{O}(d)$. La fibre de $\mathcal{O}(d)$ au-dessus d'un point x de $\mathbb{CP}^n$ est l'espace des formes d-linéaires sur x, vu comme droite complexe de $\mathbb{C}^{n+1}$. Soit $P \in \mathbb{C}^d_{\mathrm{hom}}[X_0, \dots, X_n]$, l'espace des polynômes homogènes complexes de degré d en $n+1$ variables. Ce polynôme définit une section holomorphe globale de $\mathcal{O}(d)$ qui associe à $x \in \mathbb{CP}^n$ la restriction de P à la droite x. On obtient de la sorte toutes les sections holomorphes globales de $\mathcal{O}(d)$ (cf. [GH94, p. 194]). Ainsi, $H^0(\mathbb{CP}^n, \mathcal{O}(d)) = \mathbb{C}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ et $\mathbb{R}H^0(\mathbb{CP}^n, \mathcal{O}(d)) = \mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$.

Notation 1.3.8. Soit $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ un multi-indice, on note $|\alpha| = \alpha_0 + \cdots + \alpha_n$ sa *longueur*. On note également $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ et $\alpha! = (\alpha_0)! \ldots (\alpha_n)!$. Enfin, si $|\alpha| = d$, on note $\binom{d}{\alpha}$ le coefficient multinomial $\frac{d!}{\alpha!}$.

Lemme 1.3.9. *Les monômes X^α avec $|\alpha| = d$ forment une famille orthonogonale de $\mathbb{C}^d_{hom}[X_0, \ldots, X_n]$ pour le produit scalaire* (1.3.1)*. De plus, pour tout α de longueur d,*

$$\|X^\alpha\|^2 = \frac{\pi^n \alpha!}{(d+n)!}.$$

Démonstration. Soient $d \in \mathbb{N}^*$ et α et $\beta \in \mathbb{N}^{n+1}$ de longueur d. Soit $x \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ et $[x]$ sa classe dans $\mathbb{CP}^n$. On a $h_d\left(X^\alpha([x]), X^\beta([x])\right) = x^\alpha \overline{x}^\beta$. En particulier, cette quantité ne dépend pas du relevé de $[x]$ dans $\mathbb{S}^{2n+1}$. Comme ω_{FS} induit la métrique quotient de la sphère,

$$\begin{aligned}\left\langle X^\alpha, X^\beta \right\rangle &= \int_{\mathbb{CP}^n} h_d\left(X^\alpha([x]), X^\beta([x])\right) \frac{(\omega_{FS})^n}{n!} \\ &= \frac{1}{2\pi} \int_{x \in \mathbb{S}^{2n+1}} x^\alpha \overline{x}^\beta \, \mathrm{d}\theta(x),\end{aligned}$$

où $\mathrm{d}\theta$ est la mesure euclidienne sur la sphère. On utilise ensuite à répétition la formule :

$$2\int_0^{+\infty} r^{2\alpha+1} e^{-r^2} \, \mathrm{d}r = \alpha!$$

pour obtenir

$$\begin{aligned}\left\langle X^\alpha, X^\beta \right\rangle &= \frac{1}{\pi(d+n)!} \int_{x \in \mathbb{S}^{n+1}} x^\alpha \overline{x}^\beta r^{2n+2d+1} e^{-r^2} \, \mathrm{d}\theta(x) \, \mathrm{d}r \\ &= \frac{1}{\pi(d+n)!} \int_{z \in \mathbb{C}^{n+1}} z^\alpha \overline{z}^\beta e^{-\|z\|^2} \, \mathrm{d}z \\ &= \frac{1}{\pi(d+n)!} \prod_{j=0}^{n} \left(\int_{z \in \mathbb{C}} z_j^{\alpha_j} \overline{z_j}^{\beta_j} e^{-|z_j|^2} \, \mathrm{d}z_j \right) \\ &= \frac{\pi^n}{(d+n)!} \prod_{j=0}^{n} \delta_{\alpha_j \beta_j} (\alpha_j)!,\end{aligned}$$

où $\delta_{\alpha_j \beta_j} = 1$ si $\alpha_j = \beta_j$ et vaut 0 sinon. □

Ainsi, la famille des :

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sqrt{\binom{d}{\alpha}} X^\alpha,$$

où α parcourt les multi-indices de longueur d, est une base de $\mathbb{C}^d_{\mathrm{hom}}[X_0, \ldots, X_n]$, orthonormée pour le produit scalaire (1.3.1). C'est aussi une base orthonormée de $\mathbb{R}^d_{\mathrm{hom}}[X_0, \ldots, X_n]$ pour le produit scalaire euclidien induit.

Un vecteur gaussien standard dans $\mathbb{R}H^0(\mathbb{CP}^n, \mathcal{O}(d))$ pour ce produit scalaire est donc un polynôme homogène de degré d en $n+1$ variables de la forme :

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} \sqrt{\binom{d}{\alpha}} a_\alpha X^\alpha, \tag{1.3.3}$$

où les coefficients a_α sont des variables réelles indépendantes de loi $\mathcal{N}(0,1)$. Comme on s'intéresse au lieu des zéros de ce polynôme aléatoire, on peut oublier le facteur $\sqrt{\frac{(d+n)!}{\pi^n d!}}$. Cette distribution est appelée *distribution de Kostlan.*

On vient de voir un exemple de notre cadre algébrique réel en codimension $r=1$, lorsque $\mathcal{E}$ est trivial. En codimension supérieure, on peut aussi choisir $\mathcal{E}$ égal au fibré trivial $\mathbb{C}^r \times \mathbb{CP}^n \to \mathbb{CP}^n$, muni des structures réelle et métrique usuelle. L'espace des sections holomorphes globales réelles de $\mathcal{E} \otimes \mathcal{O}(d)$ est alors $\left(\mathbb{R}^d_{\mathrm{hom}}[X_0,\dots,X_n]\right)^r$ et les r copies de $\mathbb{R}^d_{\mathrm{hom}}[X_0,\dots,X_n]$ sont orthogonales pour (1.3.1). Un vecteur gaussien standard dans $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ est alors un r-uplet de polynômes indépendants identiquement distribués suivant une loi de Kostlan.

Remarque 1.3.10. Dans le cas de sous-variétés algébriques complexes, un vecteur gaussien complexe standard dans $H^0(\mathbb{CP}^n,\mathcal{O}(d))$ est un polynômes de $\mathbb{C}^d_{\mathrm{hom}}[X_0,\dots,X_n]$ de la forme (1.3.3) où les a_α sont des variables complexes indépendantes de loi $\mathcal{N}_\mathbb{C}(0,1)$. Ces polynômes aléatoires sont dit $U(n+1)$*-invariants* (en anglais "$SU(n+1)$ polynomials") et leur distribution est l'unique mesure gaussienne sur $\mathbb{C}^d_{\mathrm{hom}}[X_0,\dots,X_n]$ invariante par l'action naturelle de $U(n+1)$ par précomposition (voir [Kos02, thm. 5.1]).

En particulier, cette mesure est plus naturelle géométriquement qu'une distribution où les a_α seraient choisis gaussiens standards indépendants, comme dans le cas des polynômes de Kac (voir section 1.1.1). Pour des coefficients identiquement distribués, la distribution n'est pas indépendante du choix d'une base orthonormée de $\mathbb{C}^{n+1}$.

Dans le cas réel, la mesure de Kostlan est invariante sous l'action de $O(n+1)$ sur $\mathbb{R}^d_{\mathrm{hom}}[X_0,\dots,X_n]$ par précomposition mais ce n'est pas l'unique mesure gaussienne ayant cette propriété. Il existe une famille à paramètres de telles mesures, qui sont décrites dans [Kos02, sect. 5.5].

Les polynômes aléatoires de la forme (1.3.3) sont appelés *polynômes de Kostlan–Shub–Smale.* Le volume des sous-variétés algébriques définies par un système de r tels polynômes indépendants de degré d a été calculé par Kostlan [Kos93]. Shub et Smale [SS93] ont traité le cas d'un système de n polynômes indépendants en dimension n, de degrés éventuellement différents. La caractéristique d'Euler de ces sous-variétés algébriques a été calculée par Podkorytov [Pod01] dans le cas des hypersurfaces, et par Bürgisser [Bü07] en codimension supérieure. Le résultat suivant est à comparer au théorème 1.1.1.

Théorème 1.3.11 (Kostlan, 1993)**.** *Soient $d \in \mathbb{N}^*$ et $P_1,\dots,P_r \in \mathbb{R}^d_{hom}[X_0,\dots,X_n]$ des polynômes de Kostlan–Shub–Smale indépendants. Soit Z_P le lieu des zéros communs des P_i dans $\mathbb{RP}^n$, alors on a :*

$$\mathbb{E}[\mathrm{Vol}\,(Z_P)] = d^{\frac{r}{2}}\,\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right).$$

Théorème 1.3.12 (Podkorytov, 2001 – Bürgisser, 2007)**.** *Soient $d \in \mathbb{N}^*$ et $P_1,\dots,P_r$ des polynômes de Kostlan–Shub–Smale indépendants de degré d. Soit Z_P le lieu des zéros communs des P_i dans $\mathbb{RP}^n$, alors si $n-r$ est pair on a :*

$$\mathbb{E}[\chi(Z_p)] = d^{\frac{r}{2}} \sum_{p=0}^{\frac{n-r}{2}} (1-d)^p \frac{\Gamma\left(p+\frac{r}{2}\right)}{p!\Gamma\left(\frac{r}{2}\right)},$$

où Γ est la fonction gamma d'Euler.

1.3.3 Énoncés des résultats

Nos deux premières contributions à l'étude des sous-variétés algébriques aléatoires étendent les théorèmes 1.3.11 et 1.3.12. Ce sont les équivalents dans ce cadre de nos théorèmes 1.2.5 et 1.2.6.

Théorème 1.3.13. *Soit $\mathcal{X}$ une variété projective complexe de dimension n, munie d'une structure réelle. Soit M le lieu réel de $\mathcal{X}$, supposé non vide. Soient $\mathcal{E} \to \mathcal{X}$ un fibré hermitien de rang $r \in \{1, \dots, n\}$ et $\mathcal{L} \to \mathcal{X}$ un fibré en droites hermitien positif, tous deux équipés de structures réelles compatibles. Soient enfin $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien standard et $|\mathrm{d}V_d|$ la mesure riemannienne d'intégration sur le lieu d'annulation réel de s_d. Alors, pour tout $\phi \in \mathcal{C}^0(M)$, on a :*

$$\mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi \rangle] = d^{\frac{r}{2}} \left(\int_M \phi \, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} + \|\phi\|_\infty \, O\!\left(d^{\frac{r}{2}-1}\right),$$

lorsque $d \to +\infty$. De plus le terme $O\!\left(d^{\frac{r}{2}-1}\right)$ est indépendant de ϕ.

Théorème 1.3.14. *Sous les mêmes hypothèses que le théorème 1.3.13, soit s_d un vecteur gaussien standard dans $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ et soit Z_d son lieu d'annulation réel. Si $n - r$ est pair, on a l'asymptotique suivante quand d tend vers l'infini :*

$$\mathbb{E}[\chi(Z_d)] = (-1)^{\frac{n-r}{2}} d^{\frac{n}{2}} \, \mathrm{Vol}\left(M\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right) \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \, \mathrm{Vol}\left(\mathbb{S}^{n}\right) \mathrm{Vol}\left(\mathbb{S}^{n-1}\right)} + O\!\left(d^{\frac{n}{2}-1}\right).$$

Nous calculons aussi l'asymptotique de la variance des variables aléatoires $\langle |\mathrm{d}V_d| \, , \phi \rangle$ avec $\phi \in \mathcal{C}^0(M)$, en toute codimension sauf la codimension maximale. Avant d'énoncer ce résultat il nous faut introduire quelques notations.

Définition 1.3.15. Soit $\phi \in \mathcal{C}^0(M)$, on note ϖ_ϕ son *module d'uniforme continuité*. Rappelons qu'il est défini par :

$$\begin{array}{rccc} \varpi_\phi : & (0, +\infty) & \longrightarrow & [0, +\infty) \\ & \varepsilon & \longmapsto & \sup \left\{ |\phi(x) - \phi(y)| \;\middle|\; (x, y) \in M^2, \rho_g(x, y) \leqslant \varepsilon \right\}, \end{array}$$

où $\rho_g(\cdot, \cdot)$ est la distance géodésique sur (M, g).

Définition 1.3.16. Soit $A \in \mathcal{M}_{rn}(\mathbb{R})$ une matrice de taille $r \times n$ à coefficients réels, on note $\left|\det{}^\perp(A)\right|$ le *jacobien* de A :

$$\left|\det{}^\perp(A)\right| = \sqrt{\det\left(AA^{\mathrm{t}}\right)}.$$

Définition 1.3.17. Pour tout $t \in \left]0, +\infty\right[$ on note $(X(t), Y(t))$ un vecteur gaussien centré dans $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ dont la matrice de variance est donnée par :

$$\left(\begin{array}{ccccc|ccccc}
1 - \frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} - \frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & \vdots & 0 & e^{-\frac{t}{2}} & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & 0 & \vdots & & \ddots & e^{-\frac{t}{2}} & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} \\
\hline
e^{-\frac{t}{2}} - \frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & 1 - \frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & e^{-\frac{t}{2}} & \ddots & & \vdots & 0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & e^{-\frac{t}{2}} & 0 & \vdots & & \ddots & 1 & 0 \\
0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) \otimes I_r,$$

où I_r est la matrice identité de taille r.

C'est-à-dire, en notant $X_{ij}(t)$ (respectivement $Y_{ij}(t)$) les coefficients de $X(t)$ (respectivement $Y(t)$), les couples $(X_{ij}(t), Y_{ij}(t))$ pour $1 \leqslant i \leqslant r$ et $1 \leqslant j \leqslant n$ sont globalement indépendants et la variance de $(X_{ij}(t), Y_{ij}(t))$ a pour matrice :

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & e^{-\frac{t}{2}}\left(1 - \frac{t}{1-e^{-t}}\right) \\ e^{-\frac{t}{2}}\left(1 - \frac{t}{1-e^{-t}}\right) & 1 - \frac{te^{-t}}{1-e^{-t}} \end{pmatrix} \quad \text{si } j = 1, \text{ et}$$

$$\begin{pmatrix} 1 & e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} & 1 \end{pmatrix} \qquad \text{sinon.}$$

Notation 1.3.18. On pose $\alpha_0 = \dfrac{n-r}{2(2r+1)(2n+1)}$.

Rappelons que si X et Y sont deux variables aléatoires réelles, alors leur covariance est :

$$\mathrm{Cov}(X,Y) = \mathbb{E}\Big[(X - \mathbb{E}[X])\ (Y - \mathbb{E}[Y])\Big] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]\,.$$

Théorème 1.3.19. *Soit $\mathcal{X}$ une variété projective complexe de dimension $n \geqslant 2$, munie d'une structure réelle. Soit M le lieu réel de $\mathcal{X}$, supposé non vide. Soient $\mathcal{E} \to \mathcal{X}$ un fibré hermitien de rang $r \in \{1, \dots, n-1\}$ et $\mathcal{L} \to \mathcal{X}$ un fibré en droites hermitien positif, tous deux équipés de structures réelles compatibles. Soient $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien standard et $|\mathrm{d}V_d|$ la mesure riemannienne d'intégration sur le lieu d'annulation réel de s_d.*

Soit $\beta \in \left]0, \frac{1}{2}\right[$, il existe $C_\beta > 0$ tel que pour tout $\alpha \in \left]0, \alpha_0\right[$ et pour tout $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, on ait :

$$\begin{aligned} \mathrm{Cov}(\langle |\mathrm{d}V_d|\ , \phi_1\rangle\,, \langle |\mathrm{d}V_d|\ , \phi_2\rangle) &= d^{r-\frac{n}{2}}\left(\int_M \phi_1\phi_2\,|\mathrm{d}V_M|\right)\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r}\mathcal{I}_{n,r} \\ &\quad + \|\phi_1\|_\infty \|\phi_2\|_\infty\, O\Big(d^{r-\frac{n}{2}-\alpha}\Big) + \|\phi_1\|_\infty\, \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\Big(d^{r-\frac{n}{2}}\Big)\,, \end{aligned} \tag{1.3.4}$$

quand $d \to +\infty$, où

$$\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty}\left(\frac{\mathbb{E}\left[\left|\det{}^\perp(X(t))\right|\left|\det{}^\perp(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2\right) t^{\frac{n-2}{2}}\,\mathrm{d}t < +\infty. \tag{1.3.5}$$

De plus, les termes d'erreur $O\Big(d^{r-\frac{n}{2}-\alpha}\Big)$ et $O\Big(d^{r-\frac{n}{2}}\Big)$ dans (1.3.4) *ne dépendent pas de ϕ_1 et ϕ_2.*

Le contenu de cet énoncé est discuté assez longuement au début du chapitre 3, voir notamment la remarque 3.1.8. Plutôt que de reproduire cette discussion ici, nous renvoyons à la section 3.1. Pour conclure cette section nous énonçons plusieurs corollaires du théorème 1.3.19.

Corollaire 1.3.20. *Sous les hypothèses du thm. 1.3.19, soit $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien standard et Z_d le lieu de ses zéros réels. Alors on a :*

$$\mathrm{Var}(\mathrm{Vol}\,(Z_d)) = d^{r-\frac{n}{2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r}\mathcal{I}_{n,r} + o(d^{r-\frac{n}{2}}).$$

Corollaire 1.3.21. *Sous les hypothèses du thm. 1.3.19, soient* $\alpha \geqslant \frac{r}{2} - \frac{n}{4}$ *et* $\phi \in \mathcal{C}^0(M)$*. Alors, pour tout* $\varepsilon > 0$*, on a :*

$$\mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \ , \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \ , \phi\rangle]\right| > d^{\alpha}\varepsilon\right) = \frac{1}{\varepsilon^2} O\left(d^{r-\frac{n}{2}-2\alpha}\right),$$

où le terme d'erreur est indépendant de ε*.*

Corollaire 1.3.22. *Sous les hypothèses du thm. 1.3.19, soit* U *un ouvert de* M*, on a :*

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right),$$

quand $d \to +\infty$*.*

Notre dernier corollaire concerne une suite aléatoire de sections de degré croissant. Soit $\mathrm{d}\nu_d$ la mesure gaussienne standard sur $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, on note $\mathrm{d}\nu$ la mesure de probabilité produit $\bigotimes_{d\in\mathbb{N}} \mathrm{d}\nu_d$ sur $\prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

Corollaire 1.3.23. *Sous les hypothèses du thm. 1.3.19, supposons de plus que* $n \geqslant 3$*. Soit* $(s_d)_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ *une suite aléatoire de sections. Alors,* $\mathrm{d}\nu$*-presque surement, on a :*

$$d^{-\frac{r}{2}} \left|\mathrm{d}V_{s_d}\right| \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} \left|\mathrm{d}V_M\right|,$$

au sens de la convergence faible des mesures de Radon.

1.3.4 Résultats liés

Commençons par signaler que les résultats de Nazarov et Sodin, déjà cités dans la section 1.2.3 concernent également notre modèle algébrique réel (voir [NS15, sect. 2.2]). De même, on pourrait s'attendre à ce que le théorème 1.2.10 de Taylor et Adler possède un équivalent dans ce cadre. À notre connaissance il n'existe pas de résultats en ce sens. Un premier problème pour étendre le théorème 1.2.10 au cas algébrique vient du fait que la métrique induite par le champ dépend aussi des choix d'une connexion et d'une métrique sur le fibré $\mathcal{L}$.

Dans le cas d'un polynôme de Kostlan–Shub–Smale dans $\mathbb{RP}^1$, Dalmao [Dal15] obtient une asymptotique d'ordre $\sqrt{d}$ pour la variance du nombre de racines. Il prouve également un théorème central limite pour le nombre de racines quand le degré tend vers l'infini. Son développement asymptotique pour la variance [Dal15, prop. 3.1] est très semblable à (1.3.4). Le théorème 1.3.19 ne concerne pas le cas $r = n$, notre preuve échouant dans ce cas. Le problème est a priori purement technique et nous conjecturons qu'un résultat semblable est vrai en codimension maximale, en particulier pour un système de n polynômes de Kostlan–Shub–Smale indépendants en dimension n. Le résultat de Dalmao va dans ce sens.

Kratz et Leòn [KL01] ont étudié la longueur des courbes de niveau d'un processus gaussien centré stationnaire de variance unité, dans un carré $[-T, T] \times [-T, T]$ du plan $\mathbb{R}^2$. Ils calculent les asymptotiques de la moyenne et de la variance de cette longueur quand $T \to +\infty$. De plus, ils prouvent que cette quantité vérifie un théorème central limite quand T tend vers l'infini. Ce résultat s'applique en particulier au champ gaussien centré dont la fonction de corrélation est :

$$(x, y) \longmapsto \exp\left(-\frac{1}{2}\left\|x - y\right\|^2\right), \tag{1.3.6}$$

où $\|\cdot\|$ est la norme usuelle de $\mathbb{R}^2$. Ce champ est la limite d'échelle locale, au sens de [NS15], quand d tend vers l'infini, du champ défini par une section gaussienne standard $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, lorsque $n = 2$ et $r = 1$. Très récemment, Beffara et Gayet [BG16] ont prouvé que les courbes nodales définies par le champ limite (1.3.6) vérifient certaines propriétés de percolation, établissant ainsi un lien entre les sous-variétés aléatoires étudiées dans ce manuscrit et le monde des percolations. Un tel lien avait été conjecturé dans [BS07], mais la conjecture portait sur un modèle riemannien et non pas algébrique réel.

Le modèle algébrique réel que nous avons décrit dans la section 1.3.1 a été principalement étudié par Gayet et Welschinger. Ils obtiennent dans ce cadre les mêmes résultats que dans le cas des ondes riemanniennes aléatoires, cette fois en toute codimension (voir [GW14b, GW16] pour le cas des hypersurfaces et [GW15a] en codimension supérieure). Ils démontrent l'équivalent algébrique réel du théorème 1.2.13 en considérant des types de difféomorphismes de sous-variétés de $\mathbb{R}^n$ de codimension r, ainsi que l'équivalent du théorème 1.2.8 pour des sous-variétés algébriques aléatoires de codimension r. Ceci leur permet d'établir un encadrement asymptotique d'ordre $d^{\frac{n}{2}}$ pour les nombres de Betti moyens de Z_d.

Théorème 1.3.24 (Gayet–Welschinger, 2015)**.** *Soit $\mathcal{X}$ une variété projective complexe de dimension n, munie d'une structure réelle. Soit M le lieu réel de $\mathcal{X}$, supposé non vide. Soient $\mathcal{E} \to \mathcal{X}$ un fibré hermitien de rang $r \in \{1, \dots, n\}$ et $\mathcal{L} \to \mathcal{X}$ un fibré en droites hermitien positif, tous deux équipés de structures réelles compatibles. Soient $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien standard et Z_d son lieu d'annulation réel. Alors, pour tout $i \in \{0, \dots, n-r\}$, il existe des constantes strictement positives $c(n, r, i)$ et $C(n, r, i)$ telles que :*

$$c(n,r,i)\operatorname{Vol}(M) \leqslant \liminf_{d\to+\infty} d^{-\frac{n}{2}}\mathbb{E}[b_i(Z_d)] \leqslant \limsup_{d\to+\infty} d^{-\frac{n}{2}}\mathbb{E}[b_i(Z_d)] \leqslant C(n,r,i)\operatorname{Vol}(M),$$

où $b_i(Z_d)$ est le i-ème nombre de Betti de Z_d. De plus, les constantes $c(n, r, i)$ et $C(n, r, i)$ ne dépendent que de n, r et i et sont explicites.

En lien, avec le théorème 1.3.14, comme dans le cas riemannien, le résultat de Gayet et Welschinger [GW15a, thm. 3.1.3] concernant l'espérance du nombre de points critiques d'une fonction de Morse restreinte à Z_d permet de montrer que, quand $d \to +\infty$,

$$\mathbb{E}[\chi(Z_d)] = C_{n,r} d^{\frac{n}{2}} + o(d^{\frac{n}{2}}),$$

où $C_{n,r}$ ne dépend que de n et r. Là encore, $C_{n,r}$ pourrait être nulle. C'est d'ailleurs le cas pour $n-r$ impair. Dans le modèle algébrique réel, la valeur de la constante universelle $C_{n,r}$ est donnée par le théorème 1.3.12, en spécifiant au cas des polynômes de Kostlan–Shub–Smale. Cela fournit une démonstration alternative du théorème 1.3.14.

La première partie du 16ème problème de Hilbert porte sur l'étude du nombre de composantes connexes d'une courbe algébrique dans $\mathbb{RP}^2$ et les arrangements possibles pour ses composantes en fonction du degré d de la courbe. On dispose de bornes, dites de Harnack–Klein, qui majorent le nombre de composantes connexes d'une telle courbe en fonction de d. Dans le cas de courbes algébriques aléatoires, Gayet et Welschinger [GW11a] ont prouvé que la probabilité qu'une courbe possède le nombre maximal de composantes autorisé par les bornes de Harnack–Klein tendait vers 0 exponentiellement vite quand $d \to +\infty$.

Le modèle de sous-variétés aléatoires algébriques complexes décrit au début de la section 1.3.1 a été intensément étudié par Shiffman, Zelditch et leurs co-auteurs. Voir aussi [BSZ01, SZ02, SZ03] dans un cadre symplectique proche. Avec les mêmes notations qu'à la section 1.3.1, soit $s_d \in H^0(\mathcal{X}, \mathcal{L}^d)$ une section aléatoire de loi gaussienne complexe standard et soit $Z_d^{\mathbb{C}}$ le courant d'intégration sur le lieu des zéros complexes de s_d. Dans

[SZ99], les auteurs calculent l'asymptotique de la moyenne $\mathbb{E}\left[Z_d^{\mathbb{C}}\right]$ lorsque $d \to +\infty$, ainsi qu'une majoration asymptotique de la variance de $Z_d^{\mathbb{C}}$. Ils en déduisent un résultat d'équidistribution similaire au corollaire 1.3.23 dans le cas complexe. Profitons-en pour signaler que Zelditch obtient également un résultat semblable pour un modèle de bande passante (cf. section 1.2.3) sur une variété riemannienne analytique réelle, voir [Zel09, cor. 2].

Soit maintenant s_d une gaussienne standard complexe dans $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ où $\mathcal{E}$ est trivial, on note toujours $Z_d^{\mathbb{C}}$ le courant associé aux zéros de s_d. Dans [SZ08], Shiffman et Zelditch calculent l'asymptotique de la variance du volume de $s_d^{-1}(0) \cap U$, où U est un ouvert de $\mathcal{X}$, et dans [SZ10] ils obtiennent l'asymptotique de la variance des statistiques linéaires $\left\langle Z_d^{\mathbb{C}}\,, \phi \right\rangle$, où ϕ est une $(n-r, n-r)$-forme lisse. Dans les deux cas, la variance admet un équivalent d'ordre $d^{2r-n-\frac{1}{2}}$. Dans le cas des hypersurfaces, ils obtiennent dans [SZ10] un théorème central limite pour les statistiques linéaires $\left\langle Z_d^{\mathbb{C}}\,, \phi \right\rangle$. Ils utilisent pour cela un résultat de normalité asymptotique établit dans [ST04].

Dans [BSZ00], les auteurs calculent la fonction de corrélation à p points pour le lieu des zéros de s_d, ainsi que la limite d'échelle de cette fonction. Dans le cas $p = 2$, l'équivalent de cette fonction pour notre modèle algébrique réel apparaît dans la preuve du théorème 1.3.19. Sa limite d'échelle apparaît quant à elle dans la définition de la constante $\mathcal{I}_{n,r}$, voir (1.3.5).

Dans [SZZ08], Shiffman, Zelditch et Zrebiec prouvent que la probabilité que s_d ne s'annulent pas sur un ouvert $U \subset \mathcal{X}$ tend vers 0 exponentiellement vite lorsque $d \to +\infty$. Ce résultat est à rapprocher du corollaire 1.3.22. Citons enfin que Shiffman, Zelditch et Zhong [SZZ11] ont étudié la distribution conditionnelle de $Z_d^{\mathbb{C}}$ sachant que $s_d(x) = 0$.

1.4 Ébauches de preuves

Nous présentons maintenant les idées principales des preuves des théorèmes 1.2.5, 1.2.6, 1.3.13, 1.3.14 et 1.3.19. Les deux ingrédients essentiels de ces démonstrations sont : d'une part des estimations sur la fonction de corrélation du champ définissant nos sous-variétés aléatoires, et d'autre part une formule de Kac–Rice. Ces deux ingrédients font l'objet des sections 1.4.1 et 1.4.2 respectivement. Les sections 1.4.3, 1.4.4 et 1.4.5 présentent respectivement les idées de preuves pour : le calcul du volume moyen (thm. 1.2.5 et 1.3.13), le calcul de la caractéristique d'Euler moyenne (thm. 1.2.6 et 1.3.14) et le calcul de la variance du volume (thm. 1.3.19).

1.4.1 La fonction de corrélation

Plaçons nous dans le cadre général décrit dans la section 1.1. Soit (M, g) une variété riemannienne de dimension n et soit $r \in \{1, \dots, n\}$. Nous considérons $f \sim \mathcal{N}(0, \mathrm{Id})$ une application aléatoire dans V, où V est un sous-espace de dimension finie N de $\mathcal{C}^\infty(M, \mathbb{R}^r)$ muni d'un produit scalaire euclidien. De plus, on suppose que V est 0-ample (voir def. 1.1.2), de sorte que $Z_f = f^{-1}(0)$ est presque surement lisse de codimension r.

Le vecteur aléatoire f définit un champ gaussien centré $(f(x))_{x \in M}$. La distribution de ce champ est alors décrite par sa fonction de corrélation :

$$E : (x, y) \longmapsto \mathrm{Cov}(f(x), f(y))\,.$$

Ici, nous notons $\mathrm{Cov}(f(x), f(y))$ l'opérateur de covariance de $f(x)$ et $f(y)$ qui est une application linéaire de $\mathbb{R}^r$ dans $\mathbb{R}^r$ définie par :

$$\mathrm{Cov}(f(x), f(y)) : v \longmapsto \mathbb{E}[f(x) \langle v\,, f(y) \rangle]\,,$$

où $\langle \cdot\,, \cdot \rangle$ est le produit scalaire usuel de $\mathbb{R}^r$. Ainsi, E est une application de $M \times M$ dans $\mathrm{End}(\mathbb{R}^r) = \mathbb{R}^r \otimes (\mathbb{R}^r)^*$. Si on note $f(y)^*$ la forme linéaire $\langle \cdot\,, f(y) \rangle$ alors on a :

$$\forall x, y \in M, \qquad E(x,y) = \mathbb{E}[f(x) \otimes f(y)^*]\,. \tag{1.4.1}$$

Ce point de vue se généralise au cas où f est, non plus une fonction, mais une section d'un fibré vectoriel $\mathcal{F} \to M$ de rang r, muni d'une métrique $h_{\mathcal{F}}$. Dans ce cas, on peut toujours écrire :

$$E(x,y) = \mathrm{Cov}(f(x), f(y)) = \mathbb{E}[f(x) \otimes f(y)^*]\,,$$

pour tout $x, y \in M$, où cette fois $f(y)^* = h_{\mathcal{F}}(\cdot, f(y))$. L'opérateur de covariance $E(x,y)$ est alors une application linéaire de $\mathcal{F}_y$ dans $\mathcal{F}_x$, où on a noté $\mathcal{F}_x$ pour la fibre de $\mathcal{F}$ au-dessus de x. La "fonction" de corrélation E est donc en réalité une section du fibré $\mathcal{F} \boxtimes \mathcal{F}^* \to M \times M$ (voir sect. 3.2.3 pour plus de détails).

Notons qu'en dérivant la relation (1.4.1), par rapport à x par exemple, on obtient la covariance du couple $(d_x f, f(y))$. Par dérivations successives, les dérivées de E donnent la covariance de n'importe quel couple de dérivées de f (voir la section 3.2.3 pour plus de détails).

Lemme 1.4.1. *Soit $(f_1, \dots, f_N)$ une base orthonormée de V. Pour tout $x, y \in M$, on a :*

$$E(x,y) = \sum_{i=1}^{N} f_i(x) \otimes f_i(y)^*.$$

Démonstration. On décompose f sur la base des f_i, on a donc $f = \sum a_i f_i$ avec des coefficients a_i gaussiens standards indépendants. Puis,

$$E(x,y) = \mathbb{E}[f(x) \otimes f(y)^*] = \sum_{1 \leqslant i,j \leqslant N} \mathbb{E}[a_i a_j]\, f_i(x) \otimes f_j(y)^* = \sum_{i=1}^{N} f_i(x) \otimes f_i(y)^*. \qquad \square$$

Cette expression permet de montrer que V est 0-ample si et seulement si, pour tout $x \in M$, $E(x,x)$ est inversible. Ceci est détaillé dans le lemme 2.2.2.

Dans les deux cas qui nous intéressent, la structure euclidienne sur V vient de la restriction du produit scalaire L^2 sur $\mathcal{C}^\infty(M, \mathbb{R}^r)$ (resp. sur $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$). Rappelons que ces produits scalaires sont définis par (1.2.3) et (1.3.1). Dans ce cas, on montre que E est le noyau de la projection orthogonale sur V. Plus précisément, on obtient le résultat suivant.

Lemme 1.4.2. *Soit V un sous-espace de dimension N de $\mathcal{C}^\infty(M, \mathbb{R}^r)$ muni du produit scalaire* (1.2.3). *Pour toute application lisse $f : M \to \mathbb{R}^r$, la projection orthogonal de f sur V est :*

$$x \longmapsto \int_{y \in M} E(x,y) f(y)\, |\mathrm{d}V_M|\,.$$

Démonstration. Soit $(f_1, \dots, f_N)$ une base orthonormée de V, la projection de f sur V est

$$\sum_{i=1}^{N} \langle f\,, f_i \rangle\, f_i.$$

Par ailleurs d'après le lemme 1.4.1, on a pour tout $x \in M$:

$$\begin{aligned}\int_{y \in M} E(x,y) f(y) \left|\mathrm{d}V_M\right| &= \int_{y \in M} \left(\sum_{i=1}^{N} f_i(x) \otimes f_i(y)^* \right) f(y) \left|\mathrm{d}V_M\right| \\ &= \sum_{i=1}^{N} f_i(x) \int_{y \in M} f_i(y)^* f(y) \left|\mathrm{d}V_M\right| \\ &= \sum_{i=1}^{N} f_i(x) \left\langle f\,, f_i \right\rangle . \end{aligned}$$

□

On prouve un lemme similaire dans le cas de sections d'un fibré (cf. prop. 3.2.6). La preuve ci-dessus utilise une base orthonormée de V. La preuve de la proposition 3.2.6 donne une version intrinsèque de cette démonstration.

Dans le modèle des ondes riemanniennes aléatoires, soit $\lambda \geqslant 0$ et $f_\lambda \in (V_\lambda)^r$ un vecteur gaussien standard, la fonction de corrélation du champ gaussien centré $(f_\lambda(x))_{x \in M}$, notée E_λ, est le noyau de la projection orthogonale sur $(V_\lambda)^r$ dans $\mathcal{C}^\infty(M, \mathbb{R}^r)$. Soient N_λ la dimension de V_λ et $(\varphi_1, \dots, \varphi_{N_\lambda})$ une base orthonormée de V_λ, on note $e_\lambda : M \times M \to \mathbb{R}$ la *fonction spectrale* du laplacien :

$$e_\lambda : (x,y) \longmapsto \sum_{i=1}^{N_\lambda} \varphi_i(x) \varphi_i(y).$$

Cette fonction ne dépend pas de la base orthonormée choisie (en particulier on peut utiliser une base de fonctions propres) et on a (voir sect. 2.2.5) :

$$E_\lambda = e_\lambda \operatorname{Id}_{\mathbb{R}^r} . \tag{1.4.2}$$

On dispose d'estimations sur le comportement de e_λ et de ses dérivées le long de la diagonale $\{(x,y) \in M^2 \mid x = y\}$ lorsque $\lambda \to +\infty$. Ces estimations sont le fruit de travaux de Hörmander [Hö68] et Bin [Bin04]. Elles sont rappelées dans la section 2.3.1. Dans le chapitre 2, nous utilisons ces estimations pour estimer E_λ et ses dérivées le long de la diagonale, ce qui nous renseigne sur l'asymptotique de la distribution de $(f_\lambda(x))_{x \in M}$.

Dans notre modèle algébrique réel, soit $d \in \mathbb{N}^*$ et $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ un vecteur gaussien standard. On note E_d la fonction de corrélation du champ gaussien centré $(s_d(x))_{x \in \mathcal{X}}$. Notons qu'on considère le champ défini par s_d sur $\mathcal{X}$ entier. Alors E_d est le noyau de la projection orthogonale sur $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ dans $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, mais c'est aussi le noyau de la projection orthogonale sur $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ dans $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$. Ce noyau est appelé *noyau de Bergman*.

On dispose là encore d'estimations précises sur le comportement du noyau de Bergman et de ses dérivées. Ces estimations sont rappelées dans la section 3.3. Elles sont tirées de [MM07, MM13, MM15]. Dans le chapitre 2, où on ne s'intéresse qu'à des calculs d'espérance, les estimations pour E_d et ses dérivées le long de la diagonale nous suffisent. En revanche, dans le chapitre 3, nous avons besoin d'informations sur le comportement hors diagonale de ce noyau. Les points essentiels sont que, d'une part, $E_d(x,y)$ possède une limite d'échelle locale quand $d \to +\infty$ qui ne dépend que de la distance (correctement renormalisée) entre x et y, et d'autre part, E_d tend vers 0 exponentiellement vite hors de la diagonale. Pour le second point, une décroissance en $O(d^{-\frac{n}{2}-1})$ suffirait, voir la section 3.4.3.

Signalons que Shiffman et Zelditch utilisent un point de vue légèrement différent qui les amènent à considérer le noyau de Szegö, au lieu du noyau de Bergman, dans leur étude des sous-variétés algébriques complexes aléatoires. Une part importante des papiers

[SZ99, SZ08, SZ10] est consacrée à établir des estimations sur ce noyau de Szegö et ses dérivées. Voir aussi [Zel98, BSZ00] et [Cat99].

Nous concluons cette section par un exemple. Dans le cas d'ondes riemanniennes aléatoires sur le cercle euclidien $\mathbb{S}^1$ (voir sect. 1.2.1), nous avons vu que V_λ était l'espace des polynômes trigonométriques de degré inférieur à $\sqrt{\lambda}$. La fonction spectrale e_λ est donc le noyau de la projection orthogonale dans $\mathcal{C}^\infty(\mathbb{S}^1)$ qui envoie une fonction sur la somme partielle de degré $\lfloor\sqrt{\lambda}\rfloor$ de sa série de Fourier. C'est donc le noyau de Dirichlet :

$$e_\lambda : (x,y) \longmapsto \frac{1}{2\pi}\frac{\sin\left((\lfloor\sqrt{\lambda}\rfloor + \frac{1}{2})(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)}.$$

Dans le modèle algébrique, on peut également expliciter complètement le noyau de Bergman dans le cas particulier des polynômes de Kostlan–Shub–Smale (voir sect. 1.3.2). C'est ce que nous ferons dans la section 2.6.2.

1.4.2 Les formules de Kac–Rice

Comme nous l'avons déjà évoqué, les formules de Kac–Rice sont un outil pour calculer les moments de quantités locales associées à nos sous-variétés aléatoires (ou plus généralement associées aux lignes de niveau d'un processus gaussien). C'est d'ailleurs ainsi que sont définies les quantités locales dans [Ana16].

Plutôt qu'un théorème unique, on dispose d'un principe général permettant d'obtenir une formule de type Kac–Rice. Ces formules peuvent varier selon les quantités auxquelles on s'intéresse et surtout selon l'ordre des moments que l'on cherche à calculer. Cela oblige à redémontrer une telle formule chaque fois que l'on change de contexte. C'est ce qui est fait dans tous les papiers cités dans cette thèse qui s'intéressent au calcul de moments d'une quantité locale (volume, caractéristique d'Euler, statistiques linéaires, ...) à l'exception de [SZ99], où les auteurs utilisent la formule de Poincaré–Lelong et le théorème de plongement asymptotiquement isométrique de Tian [Tia90]. Pour plus de détails concernant les formules de Kac–Rice, notamment pour le calcul de moments d'ordre supérieur à 2 ou pour des processus gaussiens plus généraux que ceux que nous considérons, nous renvoyons à [AW09, chap. 6] et [TA07, chap. 11].

Commençons par donner une formule de Kac–Rice qui nous servira dans les calculs d'espérances. On se place dans le cadre d'un espace vectoriel 0-ample V, de dimension N, qui est un sous-espace de $\mathcal{C}^\infty(M,\mathbb{R}^r)$, où M est une variété riemannienne. On suppose de plus que V est muni du produit scalaire (1.2.3). Soit $f \sim \mathcal{N}(0,\mathrm{Id})$ dans V, rappelons que Z_f désigne le lieu d'annulation de f, qui est presque surement lisse de codimension r et que $|\mathrm{d}V_f|$ désigne la mesure riemannienne sur Z_f.

Suivant [SS93], on considère la *variété d'incidence* Σ définie par :

$$\Sigma = \{(f,x) \in V \times M \mid f(x) = 0\}.$$

Cet ensemble est le lieu d'annulation de la fonction $F : (f,x) \mapsto f(x)$ dont la différentielle partielle par rapport à la première variable est exactement ev_x (voir def. 1.1.2). L'hypothèse de 0-amplitude de V garantit que F est une submersion, donc que Σ est une sous-variété de $V \times M$ de codimension r. Les valeurs critiques de la projection de Σ sur V sont exactement les f qui ne s'annulent pas transversalement. Le théorème de Sard montre alors que l'ensemble de ces fonctions est négligeable dans V (voir section 2.2.2 pour plus de détails).

On munit Σ de la restriction de la métrique produit sur $V \times M$ et de la mesure riemannienne induite par cette métrique. On obtient alors le résultat suivant, démontré dans l'appendice 2.C. Voir aussi les théorèmes 2.5.3 et 3.4.1.

Theorem 1.4.3 (Formule de Kac–Rice 1)**.** *Soit $\Phi : \Sigma \to \mathbb{R}$ une fonction mesurable, si Φ est positive ou si $(f,x) \mapsto \Phi(f,x)e^{-\frac{1}{2}\|f\|^2}$ est intégrable on a :*

$$\mathbb{E}\left[\int_{x\in Z_f} \Phi(f,x)\left|\mathrm{d}V_f\right|\right] = \frac{1}{(2\pi)^{\frac{r}{2}}}\int_{x\in M} \frac{\mathbb{E}\Big[\Phi(f,x)\left|\det{}^{\perp}(d_xf)\right| \Big| f(x)=0\Big]}{\left|\det{}^{\perp}(\mathrm{ev}_x)\right|}\left|\mathrm{d}V_M\right|,$$

où le numérateur de l'intégrand dans le membre de droite est l'espérance conditionnelle de $\left|\det^{\perp}(d_xf)\right|\Phi(f,x)$ sachant que $f(x)=0$.

Rappelons que ev_x est défini par déf. 1.1.2 et que le jacobien $\left|\det^{\perp}(\cdot)\right|$ est défini à la section 1.3.3 (déf. 1.3.16), voir aussi la définition 3.1.3. La 0-amplitude de V garantit que le dénominateur ne s'annule pas dans le membre de droite l'expression ci-dessus.

Remarques 1.4.4.

- La démonstration de cette formule repose sur la formule de coaire de Federer [Fed96] appliquée deux fois, pour les projections de Σ sur V et sur M. Voir l'appendice 2.C pour plus de détails.
- Cette preuve nécessite que le champ $(f(x))_{x\in M}$ vienne d'un vecteur aléatoire f dans un espace de dimension finie. Elle n'est pas valable pour un processus gaussien général.
- Classiquement, l'hypothèse sur le champ $(f(x))_{x\in M}$ pour avoir une formule de Kac–Rice est que, pour tout $x \in M$, la distribution de $f(x)$ soit non dégénérée. Pour $x\in M$, l'opérateur de variance de $f(x)$ est exactement $\mathrm{ev}_x(\mathrm{ev}_x)^*$, donc la distribution de $f(x)$ est non dégénérée si et seulement si $\left|\det^{\perp}(\mathrm{ev}_x)\right| \neq 0$, i.e. si et seulement si ev_x est surjective (voir section 3.4.1). La non dégénérescence de $f(x)$ pour tout $x\in M$ est donc équivalente à la 0-amplitude de V.
- On dispose d'une formule équivalente dans le cas où V est un espace de sections, voir le théorème 3.4.1.

Passons à une seconde formule de Kac–Rice qui apparaitra dans les calculs de variance du chapitre 3. Notons Δ la diagonale dans $M\times M$. Pour établir cette formule nous avons besoin de faire l'hypothèse supplémentaire que, pour tout $(x,y)\in M^2\setminus\Delta$, l'application d'évaluation :

$$\begin{array}{rccc} \mathrm{ev}_{x,y}: & V & \longrightarrow & \mathbb{R}^r\times\mathbb{R}^r \\ & g & \longmapsto & (g(x),g(y)) \end{array}$$

est surjective. De façon équivalente, cela signifie que pour tout $(x,y)\in M^2\setminus\Delta$, la distribution de $(f(x),f(y))$ est non dégénérée. Alors, par le même raisonnement que précédemment, on montre que :

$$\Sigma' = \left\{(f,x,y)\in V\times\left(M^2\setminus\Delta\right) \;\middle|\; f(x)=0=f(y)\right\}$$

est une sous-variété lisse de $V\times M\times M$ de codimension $2r$. On munit alors Σ' de la restriction de la métrique produit de $V\times M\times M$ et de la mesure associée à cette métrique.

Theorem 1.4.5 (Formule de Kac–Rice 2)**.** *Soit $\Phi' : \Sigma' \to \mathbb{R}$ une fonction mesurable, si Φ' est positive ou si $(f,x,y)\mapsto \Phi'(f,x,y)e^{-\frac{1}{2}\|f\|^2}$ est intégrable on a :*

$$\mathbb{E}\left[\int_{(x,y)\in Z_d^2\setminus\Delta} \Phi'(f,x,y)\left|\mathrm{d}V_f\right|^2\right] = \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta}\frac{1}{\left|\det^{\perp}(\mathrm{ev}_{x,y})\right|}\times$$
$$\mathbb{E}\Big[\Phi'(f,x,y)\left|\det{}^{\perp}(d_xf)\right|\left|\det{}^{\perp}(d_yf)\right| \Big| f(x)=0=f(y)\Big]\left|\mathrm{d}V_M\right|^2,$$

où $\left|\mathrm{d}V_f\right|^2$ et $\left|\mathrm{d}V_M\right|^2$ sont les mesures produits sur $(Z_f)^2$ et M^2 respectivement.

Cette formule est démontrée dans la section 3.4.1 dans le cas particulier de sections aléatoires dans notre modèle algébrique réel. La preuve de l'énoncé général ci-dessus est la même, aux notations près.

La raison pour laquelle on considère $M^2 \setminus \Delta$, et non pas M^2, dans l'énoncé précédent est simple. Pour tout $x \in M$, la distribution de $(f(x), f(x))$ est dégénérée. De façon équivalente $\mathrm{ev}_{x,x} : V \to \mathbb{R}^r \times \mathbb{R}^r$ n'est jamais surjective. En particulier, si on considérait aussi les points de la diagonale, on ne saurait pas montrer que Σ' est bien une variété et donc on ne saurait pas démontrer le théorème 1.4.5 par cette méthode.

La dégénérescence de $\mathrm{ev}_{x,y}$ le long de la diagonale est la source de plusieurs problèmes. D'une part, dans la seconde version de la formule de Kac-Rice, le terme $\left|\det{}^\perp(\mathrm{ev}_{x,y})\right|$ tend vers 0 quand on s'approche de la diagonale. L'intégrale de droite dans le thm. 1.4.5 a donc un pôle le long de la diagonale. Cela complique grandement l'estimation de l'intégrand au voisinage de Δ (voir section 3.4.3). D'autre part, on doit travailler sur $M^2 \setminus \Delta$ qui n'est pas compact. Cette perte de compacité est problématique notamment quand il s'agit de vérifier dans un cas concret que $\mathrm{ev}_{x,y}$ est surjective pour tout $(x, y) \in M^2 \setminus \Delta$. Dans le cas des sous-variétés algébriques aléatoires, on palie à ce problème en utilisant le théorème de plongement de Kodaira (voir prop. 3.4.2).

1.4.3 Espérance du volume

Nous esquissons maintenant la preuve des théorèmes 1.2.5 et 1.3.13. Pour cela nous nous plaçons dans le modèle des ondes riemanniennes aléatoires en codimension $r = 1$. Le passage à la codimension supérieure complexifie les calculs mais ne nécessite pas d'idée supplémentaire. La preuve est similaire dans le cas algébrique réel. Les preuves complètes sont données dans les sections 2.5.2 et 2.5.3.

Soit (M, g) une variété riemannienne de dimension n. Pour $\lambda \geqslant 0$, on note $f_\lambda \in V_\lambda$ un vecteur gaussien standard. On note aussi Z_λ l'hypersurface nodale aléatoire définie par f_λ et $|\mathrm{d}V_\lambda|$ la mesure riemannienne sur Z_λ induite par g. On a vu dans la section 1.2.1 que V_λ était 0-ample pour tout λ.

Soit $\phi \in \mathcal{C}^0(M)$, en appliquant la formule de Kac–Rice version 1 (thm. 1.4.3), on obtient :

$$\mathbb{E}[\langle Z_\lambda\,, \phi\rangle] = \mathbb{E}\left[\int_{x \in Z_\lambda} \phi(x)\,|\mathrm{d}V_\lambda|\right] = \frac{1}{\sqrt{2\pi}} \int_{x \in M} \phi(x) \frac{\mathbb{E}\left[\left|\det{}^\perp(d_x f_\lambda)\right| \,\middle|\, f_\lambda(x) = 0\right]}{\left|\det{}^\perp(\mathrm{ev}_x)\right|} |\mathrm{d}V_M|\,.$$

Comme $r = 1$, on a $\left|\det{}^\perp(d_x f_\lambda)\right| = \|d_x f_\lambda\|$. Par ailleurs $\left|\det{}^\perp(\mathrm{ev}_x)\right|$ est le déterminant de $\mathrm{ev}_x(\mathrm{ev}_x)^*$ qui est l'opérateur de variance de $f_\lambda(x)$. Par l'équation (1.4.2) on a donc :

$$\left|\det{}^\perp(\mathrm{ev}_x)\right| = \sqrt{\det(E_\lambda(x, x))} = \sqrt{e_\lambda(x, x)},$$

où on rappelle que E_λ est la fonction de corrélation de $(f_\lambda(x))_{x \in M}$ et e_λ est la fonction spectrale du laplacien. Pour calculer l'asymptotique de $\mathbb{E}[\langle Z_\lambda\,, \phi\rangle]$, il s'agit donc de calculer l'asymptotique à x fixé et lorsque $\lambda \to +\infty$ de :

$$\frac{1}{\sqrt{e_\lambda(x, x)}} \mathbb{E}\left[\|d_x f_\lambda\| \,\middle|\, f_\lambda(x) = 0\right]. \tag{1.4.3}$$

L'espérance conditionnelle ci-dessus ne dépend que de e_λ et de ses dérivées le long de la diagonale. On peut donc utiliser les estimations de Hörmander et Bin (voir section 2.3.1) pour effectuer ce calcul, c'est l'objet du lemme 2.5.4. Il ne reste alors plus qu'à mener les calculs à leur terme.

Il apparaît que l'asymptotique de (1.4.3) ne dépend ni de x ni de M, mais seulement de n (et r dans le cas de codimension supérieure). C'est une conséquence directe du fait que l'asymptotique de e_λ le long de la diagonale ne dépend que de n (voir thm. 2.3.1).

1.4.4 Espérance de la caractéristique d'Euler

Passons maintenant à une esquisse de preuve des théorèmes 1.2.6 et 1.3.14. Les preuves complètes font l'objet des sections 2.5.4 et 2.5.5. Comme dans la section précédente, on se place dans le modèle des ondes riemanniennes aléatoires avec $r = 1$. Pour simplifier les écritures on suppose que $n = 3$. Ici encore, la complexité supplémentaire lorsque l'on considère n et r quelconques est purement d'ordre technique.

On conserve les notations de la section précédente. On note de plus $\kappa_\lambda(x)$ la courbure de Gauss de la surface Z_λ au point x. On a alors, par le théorème de Gauss–Bonnet :

$$\chi(Z_\lambda) = \frac{1}{2\pi} \int_{x \in Z_\lambda} \kappa_\lambda(x) \left|\mathrm{d}V_\lambda\right|,$$

dès que Z_λ est bien une surface lisse. La version de dimension supérieure de ce théorème est due à Chern (voir thm. 2.4.3). La formule de Kac–Rice version 1 (thm. 1.4.3) nous donne alors :

$$\mathbb{E}[\chi(Z_\lambda)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{x \in M} \frac{\mathbb{E}\left[\kappa_\lambda(x) \left\|d_x f_\lambda\right\| \,\middle|\, f_\lambda(x) = 0\right]}{\sqrt{e_\lambda(x,x)}} \left|\mathrm{d}V_M\right|.$$

Encore une fois, il s'agit donc de trouver l'asymptotique de l'intégrande du membre de droite, à x fixé lorsque $\lambda \to +\infty$.

La première difficulté est d'exprimer $\kappa_\lambda(x)$ uniquement en fonction de f_λ et de ses dérivées (première et seconde dans ce cas) en x. Une fois que cela est fait, on peut réécrire l'espérance conditionnelle

$$\mathbb{E}\left[\kappa_\lambda(x) \left\|d_x f_\lambda\right\| \,\middle|\, f_\lambda(x) = 0\right] \tag{1.4.4}$$

comme l'espérance sachant que $f_\lambda(x) = 0$ d'une certaine fonction du vecteur gaussien centré $(f_\lambda(x), d_x f_\lambda, \nabla_x^2 f_\lambda)$, où ∇^2 est la différentielle seconde induite par la connexion de Levi–Civita. Cette quantité ne dépend donc que des valeurs de e_λ et de ses dérivées en (x,x), valeurs que l'on sait estimer grâce aux résultats de Hörmander et Bin. La seconde difficulté est alors d'ordre calculatoire, il faut mener à bout le calcul du terme dominant dans l'espérance conditionnelle (1.4.4) ci-dessus.

Revenons sur le premier point. Une formule de Gauss nous permet d'écrire :

$$\kappa_\lambda(x) = K(T_x Z_\lambda) + \frac{1}{\left\|d_x f_\lambda\right\|^2} \det(\nabla_x^2 f_\lambda), \tag{1.4.5}$$

où $K(T_x Z_\lambda)$ est la courbure sectionnelle de M évaluée sur $T_x Z_\lambda$. On utilise pour établir cette formule le fait que la seconde forme fondamentale de Z_λ en x soit $\frac{1}{\|d_x f_\lambda\|} \nabla_x^2 f_\lambda$ (voir lemme 2.4.8). Cette formule est relativement simple ici car Z_λ est une surface. Établir une formule similaire en dimension supérieure se révèle assez ardu. C'est l'objet de la section 2.4, dont le point culminant est la proposition 2.4.9.

La formule (1.4.5) ne remplit pas totalement l'objectif que nous avions fixé, à savoir exprimer κ_λ uniquement en fonction de f_λ et de ses dérivées en x. Il se trouve que le terme parasite venant de la courbure sectionnelle ne nous pose pas de problème. Cela vient du fait que, M étant compacte, ce terme est borné par une constante ne dépendant que de (M,g). On montre alors (voir lemme 2.5.6) que, dans le calcul de l'asymptotique de l'espérance conditionnelle (1.4.4), la contribution du terme venant de $K(T_x Z_\lambda)$ est d'ordre strictement plus petit que celle du terme venant de $\frac{1}{\|d_x f_\lambda\|^2} \det(\nabla_x^2 f_\lambda)$. En d'autres termes, asymptotiquement on ne perçoit plus l'influence de la courbure de la variété ambiante dans la courbure de Z_λ, au moins en moyenne. L'asymptotique de (1.4.4) est alors la même que celle de :

$$\mathbb{E}\left[\frac{1}{\|d_x f_\lambda\|}\det(\nabla_x^2 f_\lambda)\middle| f_\lambda(x)=0\right],$$

et nous savons estimer cette quantité grâce aux estimations sur e_λ et ses dérivées.

1.4.5 Variance du volume

Pour finir, nous décrivons la preuve du théorème 1.3.19. Cette preuve fait l'objet de la section 3.4. Nous nous plaçons cette fois dans le cadre de sous-variétés algébriques aléatoires.

Soit $\mathcal{X}$ une variété projective réelle de dimension n définie sur les réels et soit M son lieu réel, supposé non vide. Soient $\mathcal{E} \to \mathcal{X}$ un fibré hermitien de rang $r \in \{1,\dots,n-1\}$ et $\mathcal{L} \to \mathcal{X}$ un fibré ample en droites complexes, tous deux munis de structures réelles compatibles. On note s_d un vecteur gaussien standard dans $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ et $|\mathrm{d}V_d|$ la mesure d'intégration sur le lieu d'annulation réel de s_d. Soient ϕ_1 et $\phi_2 \in \mathcal{C}^0(M)$, nous voulons calculer l'asymptotique de :

$$\operatorname{Cov}(\langle|\mathrm{d}V_d|\ ,\phi_1\rangle\,,\langle|\mathrm{d}V_d|\ ,\phi_2\rangle) = \mathbb{E}[\langle|\mathrm{d}V_d|\ ,\phi_1\rangle\,\langle|\mathrm{d}V_d|\ ,\phi_2\rangle] - \mathbb{E}[\langle|\mathrm{d}V_d|\ ,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\ ,\phi_2\rangle]\,. \tag{1.4.6}$$

Nous commençons par prouver que, pour tout d assez grand, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ est 0-ample (voir cor. 2.3.10) et vérifie : $\forall x,y \in M^2\setminus\Delta$, $\mathrm{ev}_{x,y}$ est surjective (voir prop. 3.4.2). Le premier point est prouvé par des estimations sur le noyau de Bergman de $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Le second point pourrait être établi par une méthode semblable mais nous préférons donner une preuve moins technique utilisant le théorème de plongement de Kodaira. Une fois ces deux points établis, nous pouvons utiliser les formules de Kac–Rice de la section 1.4.2.

Notant E_d le noyau de Bergman de $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, la formule de Kac–Rice version 1 (thm. 1.4.3) donne :

$$\mathbb{E}[\langle|\mathrm{d}V_d|\ ,\phi_i\rangle] = \frac{1}{(2\pi)^{\frac{r}{2}}}\int_{x\in M}\phi_i(x)\frac{\mathbb{E}\left[\left|\det{}^\perp(\nabla_x^d s_d)\right|\,\middle|\, s_d(x)=0\right]}{\det(E_d(x,x))^{\frac{1}{2}}}\,|\mathrm{d}V_M|\,,$$

comme dans les calculs d'espérances. Ensuite la formule de Kac–Rice version 2 (thm. 1.4.5) donne :

$$\begin{aligned}\mathbb{E}[\langle|\mathrm{d}V_d|\ ,\phi_1\rangle\,\langle|\mathrm{d}V_d|\ ,\phi_2\rangle] = \frac{1}{(2\pi)^r}&\int_{(x,y)\in M^2\setminus\Delta}\frac{\phi_1(x)\phi_2(y)}{|\det^\perp(\mathrm{ev}_{x,y})|}\times\\ &\mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\nabla_y^d s_d\right)\right|\,\middle|\, s_d(x)=0=s_d(y)\right]|\mathrm{d}V_M|^2\,.\end{aligned}$$

Nous renvoyons à la section 3.4.2 pour les détails de ces calculs. Ceci nous permet d'écrire :

$$\operatorname{Cov}(\langle|\mathrm{d}V_d|\ ,\phi_1\rangle\,,\langle|\mathrm{d}V_d|\ ,\phi_2\rangle) = \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta}\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\,, \tag{1.4.7}$$

où $\mathcal{D}_d$ est défini par (3.4.9). Notons que $\mathcal{D}_d(x,y)$ ne dépend que des valeurs de E_d et de ses dérivées (première et seconde) en (x,x), (y,y), (x,y) et (y,x). Notons aussi que $\mathcal{D}_d$ possède un pôle le long de la diagonale Δ.

Remarque 1.4.6. Dans les expressions ci-dessus, ∇^d désigne une connexion réelle sur $\mathcal{E}\otimes\mathcal{L}^d$. La fonction $\mathcal{D}_d$ ne dépend pas du choix de cette connexion. Les problèmes liés au choix de ∇^d sont discutés en détails dans le chapitre 3.

On découpe maintenant M^2 en deux domaines. Notons

$$\Delta_d = \left\{ (x,y) \in M^2 \;\middle|\; \rho_g(x,y) < K\frac{\ln d}{\sqrt{d}} \right\},$$

où ρ_g est la distance géodésique dans (M,g) et la valeur de $K > 0$ sera fixée plus loin. Le choix de l'échelle $\frac{\ln d}{\sqrt{d}}$ vient de la forme du noyau de Bergman, qui fait apparaître une échelle caractéristique $\frac{1}{\sqrt{d}}$, au-delà de laquelle $\frac{1}{d^n}E_d(x,y)$ tend vers 0 lorsque d tend vers $+\infty$. Pour plus de précisions, voir les estimations de la section 3.3. Sur $M^2 \setminus \Delta_d$, le noyau de Bergman et ses dérivées (correctement renormalisés) tendent uniformément vers 0 lorsque $d \to +\infty$. Le choix de la constante K permet d'influer sur la vitesse de cette convergence.

L'expression 3.4.9 montre que $\mathcal{D}_d$ est la différence de deux termes. Le premier vient du terme $\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1\rangle \langle |\mathrm{d}V_d|, \phi_2\rangle]$ dans l'équation (1.4.6) et le second vient du terme $\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_1\rangle]\,\mathbb{E}[\langle |\mathrm{d}V_d|, \phi_2\rangle]$ de la même équation. Loin de la diagonale, i.e. sur $M^2 \setminus \Delta_d$, la différence entre ces termes tend uniformément vers 0 quand $d \to +\infty$. Le choix de K permet de choisir la vitesse de convergence. Plus précisément, pour un bon choix de K, on montre (prop. 3.4.12) que :

$$\int_{(x,y)\in M^2\setminus\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 = \|\phi_1\|_\infty \|\phi_2\|_\infty\, O(d^{r-\frac{n}{2}-1}).$$

Par ailleurs, sur $\Delta_d \setminus \Delta$, les estimations sur E_d et ses dérivées permettent de montrer qu'il existe une fonction $\mathcal{D} :]0,+\infty[\to \mathbb{R}$ telle que, pour tout $x \in M$ et tout $z \in T_xM$ tel que $\|z\| \leqslant K \ln d$,

$$\frac{1}{d^r}\mathcal{D}_d\left(x, x+\frac{z}{\sqrt{d}}\right) \xrightarrow[d\to+\infty]{} \mathcal{D}\left(\|z\|^2\right),$$

où on note abusivement $x + \frac{z}{\sqrt{d}}$ pour $\exp_x\left(\frac{z}{\sqrt{d}}\right)$. En s'autorisant quelques imprécisions, un changement de variable donne :

$$\int_{(x,y)\in\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 \simeq \int_{x\in M}\left(\int_{z\in B_{T_xM}(0,K\ln d)} \phi_1(x)\phi_2\left(x+\frac{z}{\sqrt{d}}\right)\mathcal{D}_d\left(x,x+\frac{z}{\sqrt{d}}\right)\frac{\mathrm{d}z}{d^{\frac{n}{2}}}\right)|\mathrm{d}V_M|.$$

Lorsque $d \to +\infty$, l'intégrand est équivalent à :

$$d^{r-\frac{n}{2}}\phi_1(x)\phi_2(x)\mathcal{D}\left(\|z\|^2\right),$$

ce qui permet de montrer que :

$$\int_{(x,y)\in\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 = d^{r-\frac{n}{2}}\left(\int_M \phi_1\phi_2\, |\mathrm{d}V_M|\right)\int_{z\in\mathbb{R}^n}\mathcal{D}\left(\|z\|^2\right)\mathrm{d}z + o(d^{r-\frac{n}{2}}).$$

Pour conclure la preuve, il ne reste alors plus qu'à calculer la valeur de :

$$\int_{z\in\mathbb{R}^n}\mathcal{D}\left(\|z\|^2\right)\mathrm{d}z.$$

La difficulté principale pour transformer cette ébauche de preuve en démonstration rigoureuse est de nature technique. En particulier, nous avons besoin d'une forme d'uniformité en (x,y) (resp. en (x,z)) dans les estimations ci-dessus. Cette uniformité se révèle difficile à obtenir du fait du pôle de $\mathcal{D}_d$ le long de Δ.

1.5 Organisation du manuscrit

La suite de ce manuscrit est constituée de deux chapitres. Le chapitre 2 est la reproduction de l'article [Let16a]. Il traite de nos résultats concernant le volume moyen et la caractéristique d'Euler moyenne, dans le cadre des ondes riemanniennes aléatoires et dans notre modèle algébrique réel (thm. 1.2.5, 1.2.6, 1.3.13 et 1.3.14). Le chapitre 3 est la reproduction de la prépublication [Let16b], soumise en août 2016. Son objet est la démonstration du théorème 1.3.19 concernant la variance du volume de sous-variétés algébriques réelles.

Les articles [Let16a, Let16b] sont reproduits dans les chapitres 2 et 3 tels qu'ils ont été respectivement publié et soumis, à la correction de quelques fautes de frappe près. De ce fait, les sections 2.1 et 2.2 d'une part, et 3.1 et 3.2 d'autre part, reprennent largement ce que nous venons de dire dans le chapitre 1. Elles s'intersectent aussi fortement.

La différence majeure entre les chapitres 2 et 3 est le point de vue adopté sur la fonction de corrélation, et donc sur le noyau de Bergman (rappelons que le chapitre 3 ne traite que du cas algébrique). Dans le chapitre 3, $E_d(x, y)$ est défini comme un opérateur entre espaces euclidiens. C'est aussi le point de vue adopté dans l'introduction (chap. 1). Dans [Let16a] et donc dans le chapitre 2, $E_d(x, y)$ est défini comme une forme bilinéaire. Les deux points de vue se correspondent naturellement, via les produits scalaires euclidiens. Voir $E_d(x, y)$ comme un opérateur se révèle plus simple à l'usage. Cela simplifie notamment les écritures, en évitant le recours systématique à des écritures matricielles, et donc à des choix de bases. On pourra par exemple comparer la proposition 2.3.8 et le corollaire 3.3.8 qui présentent le même résultat, avec des points de vue différents sur la nature de E_d.

Signalons également que, dans le chapitre 2, nous établissons des estimations diagonales sur le noyau de Bergman et ses dérivées par une méthode utilisant les sections-pics de Hörmander–Tian (voir la section 2.3). Ces estimations sont suffisantes pour établir nos résultats concernant des calculs d'espérances. Dans le chapitre 3, nous reprenons les estimations établies par Ma et Marinescu [MM07] pour le noyau de Bergman (voir sect. 3.3). Les estimations au voisinage de la diagonale (sect. 3.3.2) sont exprimées dans une trivialisation spécifique du fibré $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. Elles sont établies dans [MM07, thm. 4.2.1] dans le cas où $\mathcal{X}$, $\mathcal{E}$ et $\mathcal{L}$ ne sont pas munis de structures réelles. Une part de notre travail a été de vérifier que la trivialisation utilisée dans [MM07] est bien compatible avec les structures réelles que nous ajoutons (voir section 3.3.1). Cela se révèle important dans la preuve du théorème 1.3.19 (cf. sect. 3.4.3), car cela permet d'affirmer qu'une connexion qui est triviale dans cette trivialisation est une connexion réelle (voir def. 3.3.2).

Détaillons maintenant le contenu des chapitres 2 et 3, au-delà des sections 2.1, 2.2, 3.1 et 3.2 déjà évoquées.

Dans la section 2.3, on énonce les estimations de Hörmander et Bin concernant la fonction spectrale du laplacien et on établit des estimations similaires pour le noyau de Bergman par une méthode utilisant les sections-pics de Hörmander–Tian. La section 2.4 établit une formule intégrale pour la caractéristique d'Euler d'une sous-variété définie comme le lieu d'annulation d'une submersion globale (voir prop. 2.4.9). Cette formule est une variation sur la formule de Chern–Gauss–Bonnet. La section 2.5 contient les démonstrations des théorèmes 1.2.5, 1.2.6, 1.3.13 et 1.3.14. Les démonstrations sont détaillées dans le cadre des ondes riemanniennes aléatoires et esquissées dans le cas de sous-variétés algébriques aléatoires. Dans la section 2.6.1, on traite explicitement le cas du tore plat de dimension n. On obtient notamment la valeur de la caractéristique d'Euler moyenne d'une sous-variété aléatoire de codimension r de $\mathbb{T}^n$ à λ fixé, dans le modèle des ondes riemanniennes aléatoires. De même on retrouve la valeur du volume moyen à λ fixé de ces sous-variétés. Dans la section 2.6.2, on traite explicitement le cas de l'espace projectif réel et on retrouve les résultats de Kostlan (thm. 1.3.11) et Podkorytov–Bürgisser (thm. 1.3.12).

Le chapitre 2 contient trois appendices. Dans le premier (app. 2.A) on rappelle certains faits utiles concernant les vecteurs gaussiens, le second (app. 2.B) contient la preuve d'un lemme technique et le dernier (app. 2.C) présente une preuve de la formule de Kac–Rice (thm. 1.4.3) utilisant la formule de coaire de Federer.

Dans la section 3.3 on rappelle les estimations de Ma et Marinescu pour le noyau de Bergman. Le preuve du théorème 1.3.19 fait l'objet de la section 3.4. Dans la section 3.4.1, on établit la formule de Kac–Rice version 2 (thm. 1.4.5). La section 3.4.2 établit une formule intégrale pour la variance que l'on souhaite calculer. Le gros de la preuve occupe la section 3.4.3. Finalement, la section 3.5 contient les preuves des corollaires 3.1.9, 3.1.10 et 3.1.11.

Concernant les notations, le seul conflit de notation entre les chapitres 2 et 3 est le suivant : Δ est utilisé pour l'opérateur de Laplace–Beltrami dans le chapitre 2 et pour la diagonale de M^2 dans le chapitre 3. Ce problème se pose aussi dans le chapitre 1. A priori, il n'y a pas de risque de confusion.

Signalons aussi que, dans le cadre algébrique réel, le lieu réel de $\mathcal{X}$ est noté M dans les chapitres 1 et 3 mais il est noté $\mathbb{R}\mathcal{X}$ dans le chapitre 2. La notation M dans le chapitre 2 est réservée au cadre des ondes riemanniennes aléatoires. De même, l'application d'évaluation en un point, notée ev_x dans les chapitres 1 et 3, est notée j_x^0 dans le chapitre 2. Ceci mis à part, les notations sont cohérentes tout au long du manuscrit.

Concernant les références, les références à [Let16a] dans le chapitre 3 sont à comprendre comme des références au chapitre 2. Les numéros d'énoncés indiqués dans ces références sont ceux de ce manuscrit.

CHAPITRE 2

Volume moyen et caractéristique d'Euler moyenne de sous-variétés aléatoires

Ce chapitre est consacré à l'étude du volume moyen et de la caractéristique d'Euler moyenne de sous-variétés aléatoires, dans le modèle des ondes riemanniennes aléatoires décrit à la section 1.2 et dans le modèle de sous-variétés algébriques réelles décrit à la section 1.3. Il contient les preuves des théorèmes 1.2.5, 1.2.6, 1.3.13 et 1.3.14.

Ce chapitre est la reproduction de l'article [Let16a] :
T. Letendre, *Expected volume and Euler characteristic of random submanifolds*,
J. Funct. Anal., **270** (2016), no. 8, 3047–3110, doi: 10.1016/j.jfa.2016.01.007.

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2.1 Introduction

Zeros of random polynomials were first studied by Bloch and Pòlya [BP32] in the early 30s. About ten years later, Kac [Kac43] obtained a sharp asymptotic for the expected number of real zeros of a polynomial of degree d with independent standard Gaussian coefficients, as d goes to infinity. This was later generalized to other distributions by Kostlan in [Kos93]. In particular, he introduced a normal distribution on the space of homogeneous polynomials of degree d — known as the Kostlan distribution — which is more geometric, in the sense that it is invariant under isometries of $\mathbb{CP}^1$. Bogomolny, Bohigas and Leboeuf [BBL96] showed that this distribution corresponds to the choice of d independent roots, uniformly distributed in the Riemann sphere.

In higher dimension, the question of the number of zeros can be generalized in at least two ways. What is the expected volume of the zero set? And what is its expected Euler characteristic? More generally, one can ask what are the expected volume and Euler characteristic of a random submanifold obtained as the zero set of some Gaussian field on a Riemannian manifold. In this paper, we provide an asymptotic answer to these questions in the case of Riemannian random waves and in the case of real algebraic manifolds.

Let us describe our frameworks and state the main results of this paper. See section 2.2 for more details. Let (M, g) be a closed (that is compact without boundary) smooth Riemannian manifold of positive dimension n, equipped with the Riemannian measure $|\mathrm{d}V_M|$ associated to g (defined below (2.2.1)). This induces a L^2-inner product on $\mathcal{C}^\infty(M)$ defined by:

$$\forall \phi, \psi \in \mathcal{C}^\infty(M), \qquad \langle \phi\,, \psi \rangle = \int_{x \in M} \phi(x)\psi(x)\,|\mathrm{d}V_M|\,. \tag{2.1.1}$$

It is well-known that the subspace $V_\lambda \subset \mathcal{C}^\infty(M)$ spanned by the eigenfunctions of the Laplacian associated to eigenvalues smaller than λ has finite dimension. Let $1 \leqslant r \leqslant n$ and let $f^{(1)}, \dots, f^{(r)} \in V_\lambda$ be independent standard Gaussian vectors, then we denote by Z_f the zero set of $f = (f^{(1)}, \dots, f^{(r)})$. Then, for λ large enough, Z_f is almost surely a submanifold of M of codimension r (see section 2.2 below) and we denote by $\mathrm{Vol}\,(Z_f)$ its Riemannian volume for the restriction of g to Z_f. We also denote by $\chi(Z_f)$ its Euler characteristic.

Theorem 2.1.1. *Let (M, g) be a closed Riemannian manifold of dimension n. Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ. Let $f^{(1)}, \dots, f^{(r)}$ be r independent standard Gaussian vectors in V_λ, with $1 \leqslant r \leqslant n$. Then the following holds as λ goes to infinity:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \left(\frac{\lambda}{n+2}\right)^{\frac{r}{2}} \mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O\Big(\lambda^{\frac{r-1}{2}}\Big)\,.$$

Here and throughout this paper, $\mathbb{E}[\,\cdot\,]$ denotes the mathematical expectation of the quantity between the brackets and $\mathbb{S}^n$ is, as usual, the unit Euclidean sphere in $\mathbb{R}^{n+1}$.

If $n - r$ is odd, Z_f is almost surely a smooth manifold of odd dimension. In this case, $\chi(Z_f) = 0$ almost surely. If $n - r$ is even, we get the following result.

Theorem 2.1.2. *Let (M, g) be a closed Riemannian manifold of dimension n. Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ. Let $f^{(1)}, \dots, f^{(r)}$ be r independent standard Gaussian vectors in V_λ, with $1 \leqslant r \leqslant n$. Then, if $n - r$ is even, the following holds as λ goes to infinity:*

$$\mathbb{E}[\chi\,(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{\lambda}{n+2}\right)^{\frac{n}{2}} \mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r+1})\,\mathrm{Vol}\,(\mathbb{S}^{r-1})}{\pi\,\mathrm{Vol}\,(\mathbb{S}^n)\,\mathrm{Vol}\,(\mathbb{S}^{n-1})} + O\Big(\lambda^{\frac{n-1}{2}}\Big)\,.$$

We also consider the framework of the papers by Gayet and Welschinger [GW15a, GW16], see section 2.2.6 for more details. Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension n. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{X}$ of rank r. We assume that $\mathcal{X}$, $\mathcal{L}$ and $\mathcal{E}$ are equipped with compatible real structures and that the real locus $\mathbb{R}\mathcal{X}$ of $\mathcal{X}$ is non-empty.

Let $h_\mathcal{L}$ denote a Hermitian metric on $\mathcal{L}$ with positive curvature ω and $h_\mathcal{E}$ denote a Hermitian metric on $\mathcal{E}$. Both metrics are assumed to be compatible with the real structures. Then ω is a Kähler form and it induces a Riemannian metric g and a volume form $\mathrm{d}V_\mathcal{X} = \frac{\omega^n}{n!}$ on $\mathcal{X}$. For any $d \in \mathbb{N}$, the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ is equipped with a L^2-inner product similar to (2.1.1) (see section 2.2.6).

Let $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of real global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This is a Euclidean space for the above inner product. Let s be a standard Gaussian section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, then we denote by Z_s the real part of its zero set. Once again, for d large enough, Z_s is almost surely a smooth submanifold of $\mathbb{R}\mathcal{X}$ of codimension r. Let $\mathrm{Vol}\,(Z_s)$ denote the Riemannian volume of Z_s and $\chi(Z_s)$ denote its Euler characteristic. We get the analogues of Theorems 2.1.1 and 2.1.2 in this setting.

Theorem 2.1.3. *Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let $\mathcal{L}$ be an ample holomorphic Hermitian line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle over $\mathcal{X}$, both equipped with real structures compatible with the one on $\mathcal{X}$. Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as d goes to infinity:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_s)] = d^{\frac{r}{2}}\, \mathrm{Vol}\,(\mathbb{R}\mathcal{X})\, \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O\Big(d^{\frac{r}{2}-1}\Big).$$

Theorem 2.1.4. *Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let $\mathcal{L}$ be an ample holomorphic Hermitian line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle over $\mathcal{X}$, both equipped with real structures compatible with the one on $\mathcal{X}$. Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, if $n-r$ is even, the following holds as d goes to infinity:*

$$\mathbb{E}[\chi\,(Z_s)] = (-1)^{\frac{n-r}{2}}\, d^{\frac{n}{2}}\, \mathrm{Vol}\,(\mathbb{R}\mathcal{X})\, \frac{\mathrm{Vol}\,\big(\mathbb{S}^{n-r+1}\big)\,\mathrm{Vol}\,\big(\mathbb{S}^{r-1}\big)}{\pi\,\mathrm{Vol}\,(\mathbb{S}^n)\,\mathrm{Vol}\,(\mathbb{S}^{n-1})} + O\Big(d^{\frac{n}{2}-1}\Big).$$

In the case of random eigenfunctions of the Laplacian, Theorem 2.1.1 was already known to Bérard [Bé85] for hypersurfaces. See also [Zel09, thm. 1] where Zelditch shows that, in the case of hypersurfaces,

$$\sqrt{\frac{n+2}{\lambda}}\mathbb{E}[Z_f] \xrightarrow[\lambda\to+\infty]{} \frac{\mathrm{Vol}\,\big(\mathbb{S}^{n-1}\big)}{\mathrm{Vol}\,(\mathbb{S}^n)}\, |\mathrm{d}V_M|$$

in the sense of the weak convergence of measures. He also proves a similar result in the case of band limited eigenfunctions.

Let us discuss Theorems 2.1.3 and 2.1.4 when $\mathcal{X}$ is $\mathbb{CP}^n$ with the standard real structure induced by the conjugation in $\mathbb{C}^{n+1}$, $\mathcal{E}$ is the trivial bundle $\mathcal{X} \times \mathbb{C}^r$ and $\mathcal{L} = \mathcal{O}(1)$ is the hyperplane bundle with its usual metric. Then $\mathbb{R}\mathcal{X} = \mathbb{RP}^n$ and ω is the Fubini-Study metric on $\mathbb{CP}^n$, normalized so that it is the quotient of the Euclidean metric on the sphere $\mathbb{S}^{2n+1}$. Besides, $\mathbb{R}H^0(\mathcal{X}, \mathcal{L}^d)$ is the space of real homogeneous polynomials of degree d in $n+1$ variables, and Z_s is the common real zero set of r independent such polynomials.

In this setting, Kostlan [Kos93] proved that, for any $d \geqslant 1$,

$$\mathbb{E}[\mathrm{Vol}\,(Z_s)] = d^{\frac{r}{2}}\, \mathrm{Vol}\,\big(\mathbb{RP}^{n-r}\big).$$

See also the paper [SS93] by Shub and Smale, where they compute the expected number of common real roots for a system of n polynomials in n variables. The expected Euler characteristic of a random algebraic hypersurface of degree d in $\mathbb{RP}^n$ was computed by Podkorytov [Pod01]. Both Kostlan's and Podkorytov's results were generalized by Bürgisser. In [Bü06], he computed the expected volume and Euler characteristic of a submanifold Z_s of $\mathbb{RP}^n$ defined as the common zero set of r standard Gaussian polynomials $P_1, \dots, P_r$ of degree $d_1, \dots, d_r$ respectively. In particular, when these polynomials have the same degree d and $n-r$ is even, he showed that:

$$\mathbb{E}[\chi\,(Z_s)] = d^{\frac{r}{2}} \sum_{p=0}^{\frac{n-r}{2}} (1-d)^p\, \frac{\Gamma\big(p+\frac{r}{2}\big)}{p!\,\Gamma\big(\frac{r}{2}\big)}, \tag{2.1.2}$$

where Γ denotes Euler's gamma function. Theorems 2.1.3 and 2.1.4 agree with these previous results.

Recently, Gayet and Welschinger computed upper and lower bounds for the asymptotics of the expected Betti numbers of random real algebraic submanifolds of a projective manifold, see [GW15a, GW16]. This relies on sharp estimates for the expected number of critical points of index $i \in \{0, \dots, n-r\}$ of a fixed Morse function $p : \mathbb{R}\mathcal{X} \to \mathbb{R}$ restricted to the random Z_s. More precisely, let $N_i(Z_s)$ denote the number of critical points of index i of $p_{/Z_s}$, let $\mathrm{Sym}(i, n-r-i)$ denote the open cone of symmetric matrices of size $n-r$ and signature $(i, n-r-i)$ and let $\mathrm{d}\nu$ denote the standard Gaussian measure on the space of symmetric matrices. Gayet and Welschinger show [GW15a, thm. 3.1.2] that:

$$\mathbb{E}[N_i(Z_s)] \underset{d\to+\infty}{\sim} \left(\frac{d}{\pi}\right)^{\frac{n}{2}} \mathrm{Vol}\left(\mathbb{R}\mathcal{X}\right) \frac{(n-1)!}{(n-r)!} \frac{e_{\mathbb{R}}(i, n-r-i)}{2^{r-1}\Gamma\left(\frac{r}{2}\right)}, \tag{2.1.3}$$

where $e_{\mathbb{R}}(i, n-r-i) = \displaystyle\int_{\mathrm{Sym}(i,n-r-i)} \left|\det(A)\right| \mathrm{d}\nu(A)$.

One can indirectly deduce Theorem 2.1.4 from this result and from [Bü06] in the following way. By Morse theory:

$$\mathbb{E}[\chi(Z_s)] = \sum_{i=0}^{n-r} (-1)^i \mathbb{E}[N_i(Z_s)] \underset{d\to+\infty}{\sim} C_{n,r}\ d^{\frac{n}{2}}\, \mathrm{Vol}\left(\mathbb{R}\mathcal{X}\right)$$

where $C_{n,r}$ is a universal constant depending only on n and r. Specifying to the case of $\mathbb{R}\mathbb{P}^n$, equation (2.1.2) gives the value of $C_{n,r}$. Gayet and Welschinger also proved a result similar to (2.1.3) for hypersurfaces in the case of Riemannian random waves, see [GW14a]. It gives the order of growth of $\mathbb{E}[\chi(Z_f)]$ in Theorem 2.1.2, for $r = 1$.

In their book [TA07], Taylor and Adler compute the expected Euler characteristic of the excursion sets of a centered, unit-variance Gaussian field f on a smooth manifold M. This expectation is given in terms of the Lipschitz–Killing curvatures of M for the Riemannian metric g_f induced by f, see [TA07, thm. 12.4.1]. One can deduce from this result the expected Euler characteristic of the random hypersurface $f^{-1}(0)$, always in terms of the Lipschitz-Killing curvatures of (M, g_f). It might be possible to deduce Theorems 2.1.2 and 2.1.4 from this result, in the case of hypersurfaces, when the Gaussian field $(f(x))_{x\in M}$ (resp. $(s(x))_{x\in\mathbb{R}\mathcal{X}}$) has unit variance, but one would need to estimate the Lipschitz-Killing curvatures of (M, g_f) (resp. (M, g_s)) as λ (resp. d) goes to infinity.

In a related setting, Bleher, Shiffman and Zelditch [BSZ00] computed the scaling limit of the k-points correlation function for a random complex submanifold of a complex projective manifold. See also [BSZ01] in a symplectic framework. Our proofs of Theorems 2.1.2 and 2.1.4 use the same formalism as these papers, adapted to our frameworks.

We now sketch the proofs of our main results in the Riemannian setting. The real algebraic case is similar. The first step is to express $\mathrm{Vol}\left(Z_f\right)$ (resp. $\chi(Z_f)$) as the integral of some function on Z_f. In the case of the volume this is trivial, and the answer is given by the Chern–Gauss–Bonnet theorem (see section 2.4.2 below) in the case of the Euler characteristic. Then we use the Kac–Rice formula (see Theorem 2.5.3) which allows us to express $\mathbb{E}[\mathrm{Vol}\left(Z_f\right)]$ (resp. $\mathbb{E}[\chi(Z_f)]$) as the integral on M of some explicit function that only depends on the geometry of M and on the covariance function of the smooth Gaussian field defined by f.

It turns out that the covariance function of the field associated to the r independent standard Gaussian functions $f^{(1)}, \dots, f^{(r)}$ in V_λ is given by the spectral function of the Laplacian. In the algebraic case, the covariance function is given by the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. This was already used in [BSZ00, Nic15c].

Then, our results follow from estimates on the spectral function of the Laplacian (resp. the Bergman kernel) and their derivatives (see section 2.3). In the case of random waves, the estimates we need for the spectral function were proved by Bin [Bin04], generalizing results of Hörmander [Hö68]. In the algebraic case, much is known about the Bergman kernel [BBS08, BSZ00, Cat99, MM07, Zel98] but we could not find the estimates we needed in codimension higher than 1 in the literature[1]. These estimates are established in section 2.3.3 using Hörmander–Tian peak sections. Peak sections were already used in this context in [GW15a, GW16], see also [Tia90]. The author was told by Steve Zelditch, after this paper was written, that one can deduce estimates for the Bergman kernel in higher codimension from the paper [BBS08] by Berman, Berndtsson and Sjöstrand.

This paper is organized as follows. In section 2.2 we describe how our random submanifolds are generated and the setting of the main theorems. Section 2.3 is dedicated to the estimates we need for the spectral function of the Laplacian and the Bergman kernel. In section 2.4 we derive an integral formula for the Euler characteristic of a submanifold. The main theorems are proved in section 2.5, and we deal with two special cases in section 2.6: the flat torus and the real projective space. For these examples, it is possible to compute expectations for fixed λ (resp. d) and we recover the results of Kostlan and Bürgisser. Three appendices deal respectively with: some standard results about Gaussian vectors, a rather technical proof we postponed until the end, and a derivation of the Kac–Rice formula using Federer's coarea formula.

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2.2 Random submanifolds

This section is concerned with the precise definition of the random submanifolds we consider. The first two subsections explain how we produce them in a quite general setting. The third one introduces the covariance kernel, which characterizes their distribution. We also describe the distribution induced on the bundle of 2-jets in terms of this kernel. Then we describe what we called Riemannian random waves, before explaining how to adapt all this in the real algebraic case. This kind of random submanifolds has already been considered by Bérard [Bé85], Zelditch [Zel09] and Nicolaescu [Nic15c] in the Riemannian case, and by Gayet and Welschinger in the real algebraic case, see [GW15a, GW16, GW14b]. See also [Bü06, EK95, LL15] in special cases.

2.2.1 General setting

Let (M, g) be a smooth closed manifold of positive dimension n. We denote by $|\mathrm{d}V_M|$ the Riemannian measure on M induced by g. That is, if $x = (x_1, \dots, x_n)$ are local coordinates in an open set $U \subset M$ and ϕ is a smooth function with compact support in U,

$$\int_M \phi \, |\mathrm{d}V_M| = \int_{x \in U} \phi(x) \sqrt{\det(g(x))} \, \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{2.2.1}$$

[1] At the time we wrote this paper, we were well aware of the existence of [MM07], but we were trying to deduce estimates from the diagonal asymptotic expansion [MM07, thm. 4.1.1], which does not work. We realized after this paper was published that one can deduce the estimates we needed for the Bergman kernel from the near-diagonal estimates of Ma and Marinescu [MM07, thm. 4.2.1]. We used this approach in our second paper, see Section 3.3.

From now on, we fix some $r \in \{1, \dots, n\}$ that we think of as the codimension of our random submanifolds. Let $\langle \cdot , \cdot \rangle$ denote the L^2-scalar product on $\mathcal{C}^\infty(M, \mathbb{R}^r)$ induced by $|\mathrm{d}V_M|$: for any f_1 and $f_2 \in \mathcal{C}^\infty(M, \mathbb{R}^r)$,

$$\langle f_1 , f_2 \rangle = \int_{x \in M} \langle f_1(x) , f_2(x) \rangle \, |\mathrm{d}V_M| \, , \tag{2.2.2}$$

where the inner product on the right-hand side is the standard one on $\mathbb{R}^r$.

Notation 2.2.1. Here and throughout this paper $\langle \cdot , \cdot \rangle$ will always denote the inner product on the concerned Euclidean or Hermitian space.

Let V be a subspace of $\mathcal{C}^\infty(M, \mathbb{R}^r)$ of finite dimension N. For any $f \in V$, we denote by Z_f the zero set of f. Let D denote the *discriminant locus* of V, that is the set of $f \in V$ that do not vanish transversally.

If f vanishes transversally, Z_f is a (possibly empty) smooth submanifold of M of codimension r and we denote by $|\mathrm{d}V_f|$ the Riemannian measure induced by the restriction of g to Z_f. We also denote by $\mathrm{Vol}\,(Z_f)$ the volume of Z_f and by $\chi(Z_f)$ its Euler characteristic. In the case $r = n$, when $f \notin D$, Z_f is a finite set and $|\mathrm{d}V_f|$ is the sum of the Dirac measures centered on points of Z_f.

We consider a random vector $f \in V$ with standard Gaussian distribution. That is the distribution of f admits the density function:

$$x \mapsto \frac{1}{(2\pi)^{\frac{N}{2}}} \exp\left(-\frac{1}{2} \|x\|^2\right) \tag{2.2.3}$$

with respect to the Lebesgue measure of V. Under some further technical assumptions on V (see section 2.2.2 below), f vanishes transversally almost surely. Hence, the random variables $\mathrm{Vol}\,(Z_f)$ and $\chi(Z_f)$ are well-defined almost everywhere, and it makes sense to compute their expectation.

For the convenience of the reader, we gathered the few results we need about Gaussian vectors in Appendix 2.A. We introduce some notations here and refer to Appendix 2.A for further details. In the sequel, we will denote by $X \sim \mathcal{N}(m, \Lambda)$ the fact that the random vector X is distributed according to a Gaussian with mean m and variance Λ. A standard Gaussian vector is $X \sim \mathcal{N}(0, \mathrm{Id})$. We will denote by $\mathrm{d}\nu_N$ the standard Gaussian measure on a Euclidean space of dimension N, that is the measure with density (2.2.3) with respect to the Lebesgue measure.

2.2.2 The incidence manifold

Following [Nic15c], we say that V is 0-*ample* if the map $j^0_x : f \mapsto f(x)$ is onto for every $x \in M$. From now on, we assume that this is the case and we introduce an *incidence manifold* as Shub and Smale in [SS93] (see also [GW15a, GW16]).

Let $F : (f, x) \in V \times M \mapsto f(x) \in \mathbb{R}^r$ and let $\partial_1 F$ and $\partial_2 F$ denote the partial differentials of F with respect to the first and second variable respectively. For any $(f, x) \in V \times M$,

$$\partial_1 F(f, x) = j^0_x \qquad \text{and} \qquad \partial_2 F(f, x) = d_x f. \tag{2.2.4}$$

We assumed j^0_x to be surjective for every $x \in M$, thus F is a submersion. Then $\Sigma = F^{-1}(0)$ is a smooth submanifold of codimension r of $V \times M$, called the incidence manifold, and for any $(f_0, x) \in V \times M$:

$$T_{(f_0,x)}\Sigma = \{(f, v) \in V \times T_x M \mid f(x) + d_x f_0 \cdot v = 0\}.$$

We set $\pi_1 : \Sigma \to V$ and $\pi_2 : \Sigma \to M$ the projections from Σ to each factor. A vector $f \in V$ is in the range of $d_{(f_0,x)}\pi_1$ if and only if there exists some $v \in T_xM$ such that $(f, v) \in T_{(f_0,x)}\Sigma$, that is $f(x)$ is in the range of $d_x f_0$. Since V is 0-ample, the map j^0_x is onto, and $d_x f_0$ is surjective if and only if $d_{(f_0,x)}\pi_1$ is. Thus, the discriminant locus D is exactly the set of critical values of π_1. By Sard's theorem, D has measure 0 in V, both for the Lebesgue measure and for $\mathrm{d}\nu_N$, and f vanishes transversally almost surely.

We equip Σ with the restriction of the product metric on $V \times M$. Then, whenever $f \notin D$, $(\pi_1)^{-1}(f) = \{f\} \times Z_f$ is isometric to Z_f, hence we will identify these sets. Similarly, we will identify $(\pi_2)^{-1}(x) = \ker(j^0_x) \times \{x\}$ with the subspace $\ker(j^0_x)$ of V.

2.2.3 The covariance kernel

In this subsection we introduce the Schwarz kernel and covariance function associated to our space of random functions. It turns out (see Proposition 2.2.3 below) that these objects are equal. The first to use this fact were Bleher, Shiffman and Zelditch in the case of complex projective manifolds [BSZ00] and in the case of symplectic manifolds [BSZ01]. In the setting of Riemannian random waves this was used by Zelditch [Zel09] and Nicolaescu [Nic15c].

In $\mathcal{C}^\infty(M, \mathbb{R}^r)$ equipped with the L^2-inner product (2.2.2), the orthogonal projection onto V can be represented by its *Schwartz kernel*, denoted by E. That is there exists a unique $E : M \times M \to \mathbb{R}^r \otimes \mathbb{R}^r$ such that for any smooth $f : M \to \mathbb{R}^r$, the projection of f onto V is given by:

$$x \mapsto \langle E(x,\cdot)\,, f\rangle = \int_{y\in M} \langle E(x,y)\,, f(y)\rangle\, |\mathrm{d}V_M|\,. \tag{2.2.5}$$

In the previous formula, the inner product on the right-hand side is the usual one on $\mathbb{R}^r$, acting on the second factor of $\mathbb{R}^r \otimes \mathbb{R}^r$. The kernel E has the following reproducing kernel property:

$$\forall f \in V,\ \forall x \in M, \quad f(x) = \langle E(x,\cdot)\,, f\rangle\,. \tag{2.2.6}$$

If $(f_1, \ldots, f_N)$ is any orthonormal basis of V, one can check that E is defined by:

$$E : (x,y) \mapsto \sum_{i=1}^{N} f_i(x) \otimes f_i(y). \tag{2.2.7}$$

This proves that E is smooth. Besides, for all $x \in M$, $E(x,x)$ is in the span of

$$\{\zeta \otimes \zeta \mid \zeta \in \mathbb{R}^r\},$$

and for all $\zeta \in \mathbb{R}^r$:

$$\langle E(x,x)\,, \zeta\otimes\zeta\rangle = \sum_{i=1}^{N} \langle f_i(x)\otimes f_i(x)\,, \zeta\otimes\zeta\rangle = \sum_{i=1}^{N} (\langle f_i(x)\,, \zeta\rangle)^2 \geqslant 0. \tag{2.2.8}$$

This last equation shows that we can check the 0-amplitude condition for V on its kernel.

Lemma 2.2.2. *V is 0-ample if and only if, for all $x \in M$ and all $\zeta \in \mathbb{R}^r \setminus \{0\}$, we have: $\langle E(x,x)\,, \zeta\otimes\zeta\rangle > 0$. That is if and only if $E(x,x)$ is a positive-definite bilinear form on $(\mathbb{R}^r)^*$ for any $x \in M$.*

On the other hand, the standard Gaussian vector $f \in V$ defines a smooth centered Gaussian field $(f(x))_{x\in M}$ with values in $\mathbb{R}^r$. Its distribution is totally determined by its covariance function: $(x,y) \mapsto \mathrm{Cov}(f(x), f(y))$ from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$, where $\mathrm{Cov}(f(x), f(y))$ stands for the covariance form of the random vectors $f(x)$ and $f(y)$ (cf. Appendix 2.A).

Proposition 2.2.3. *Let V be a finite-dimensional subspace of $\mathcal{C}^\infty(M,\mathbb{R}^r)$ and E its Schwartz kernel. Let $f \sim \mathcal{N}(0,\mathrm{Id})$ in V, then we have:*

$$\forall x, y \in M, \qquad \mathrm{Cov}(f(x), f(y)) = \mathbb{E}[f(x) \otimes f(y)] = E(x,y).$$

Proof. Let x and $y \in M$, then the first equality is given by Lemma 2.A.7. We will now show that $\mathbb{E}[f(x) \otimes f(y)]$ satisfies condition (2.2.5) to prove the second equality. Let $f_0 : M \to \mathbb{R}^r$ be a smooth function and $x \in M$,

$$\begin{aligned}\int_{y\in M} \langle \mathbb{E}[f(x) \otimes f(y)]\ , f_0(y)\rangle \,|\mathrm{d}V_M| &= \int_{y\in M} \mathbb{E}[\langle f(x) \otimes f(y)\, , f_0(y)\rangle]\,|\mathrm{d}V_M| \\ &= \int_{y\in M} \mathbb{E}[f(x)\,\langle f(y)\, , f_0(y)\rangle]\,|\mathrm{d}V_M| = \mathbb{E}[f(x)\,\langle f\, , f_0\rangle]\,.\end{aligned}$$

If $f_0 \in V^\perp$, this equals 0. If $f_0 \in V$, we have:

$$\mathbb{E}[f(x)\,\langle f\, , f_0\rangle] = \mathbb{E}[\langle E(x,\cdot)\, , f\rangle\,\langle f_0\, , f\rangle] = \langle E(x,\cdot)\, , f_0\rangle = f_0(x),$$

where we used the reproducing kernel property (2.2.6) both for f and f_0 and we applied Lemma 2.A.8 to $f \sim \mathcal{N}(0,\mathrm{Id})$. In both cases, $x \mapsto \mathbb{E}[f(x)\,\langle f\, , f_0\rangle]$ is the projection of f_0 onto V, which shows the second equality in Proposition 2.2.3. $\square$

This tells us that the distribution of our Gaussian field is totally determined by the Schwartz kernel E. In our cases of interest, asymptotics are known for E and its derivatives, see section 2.3 below. This is what allows us to derive asymptotics for the expectation of the volume and Euler characteristic of Z_f.

2.2.4 Random jets

Let ∇^M be the Levi-Civita connection on (M,g). For any smooth $f : M \to \mathbb{R}^r$, we denote by $\nabla^2 f = \nabla^M df$ the *Hessian* of f.

Let ∂_x (resp. ∂_y) denote the partial derivative with respect to the first (resp. second) variable for maps from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$. Likewise, we denote by $\partial_{x,x}$ (resp. $\partial_{y,y}$) the second partial derivative with respect to the first (resp. second) variable twice. As for the Hessian above, all the higher order derivatives are induced by ∇^M.

Now, let $f \sim \mathcal{N}(0,\mathrm{Id})$ in V. We will describe the distribution induced by f on the 2-jets bundle of M. Let $x \in M$, then we denote by $\mathcal{J}^k_x(\mathbb{R}^r)$ the space of k-jets of smooth functions from M to $\mathbb{R}^r$ at the point x (we will only use $k \in \{0,1,2\}$). We already defined

$$\begin{aligned} j^0_x : \mathcal{C}^\infty(M,\mathbb{R}^r) &\longrightarrow \mathbb{R}^r. \\ f &\mapsto f(x) \end{aligned}$$

We define similarly,

$$\begin{aligned} j^1_x : \mathcal{C}^\infty(M,\mathbb{R}^r) &\to \mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M) \\ f &\mapsto (f(x), d_x f) \\ \text{and} \qquad j^2_x : \mathcal{C}^\infty(M,\mathbb{R}^r) &\to \mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M \oplus \mathrm{Sym}(T^*_x M)), \\ f &\mapsto (f(x), d_x f, \nabla^2_x f) \end{aligned}$$

where $\mathrm{Sym}(T^*_x M)$ denotes the space of symmetric bilinear forms on $T^*_x M$.

The map j^2_x induces an isomorphism between $\mathcal{J}^2_x(\mathbb{R}^r)$ and the image of j^2_x. In the sequel we will identify these spaces through j^2_x. Likewise, $\mathcal{J}^1_x(\mathbb{R}^r)$ and the image of j^1_x will be identified through j^1_x.

Lemma 2.2.4. *Let V be a finite-dimensional subspace of $\mathcal{C}^\infty(M,\mathbb{R}^r)$ and E its Schwartz kernel. Let $f \sim \mathcal{N}(0,\mathrm{Id})$ in V and $x \in M$. Then $j^2_x(f) = (f(x), d_xf, \nabla^2_x f)$ is a centered Gaussian vector, and its variance form $\operatorname{Var}\big(j^2_x(f)\big)$ is characterized by:*

$$\operatorname{Var}(f(x)) = \mathbb{E}[f(x)\otimes f(x)] \ = E(x,x), \tag{2.2.9}$$

$$\operatorname{Var}(d_xf) = \mathbb{E}[\nabla_x f\otimes \nabla_x f] \ = (\partial_x\partial_y E)(x,x), \tag{2.2.10}$$

$$\operatorname{Var}\big(\nabla^2_x f\big) = \mathbb{E}\big[\nabla^2_x f\otimes \nabla^2_x f\big] = (\partial_{x,x}\partial_{y,y} E)(x,x), \tag{2.2.11}$$

$$\operatorname{Cov}(f(x), d_xf) = \mathbb{E}[f(x)\otimes \nabla_x f] \ = (\partial_y E)(x,x), \tag{2.2.12}$$

$$\operatorname{Cov}\big(f(x), \nabla^2_x f\big) = \mathbb{E}\big[f(x)\otimes \nabla^2_x f\big] = (\partial_{y,y} E)(x,x), \tag{2.2.13}$$

$$\operatorname{Cov}\big(d_xf, \nabla^2_x f\big) = \mathbb{E}\big[\nabla_x f\otimes \nabla^2_x f\big] = (\partial_x\partial_{y,y} E)(x,x). \tag{2.2.14}$$

Proof. The first equality on each line is given by Lemmas 2.A.4 and 2.A.7. Then Proposition 2.2.3 gives the second equality in (2.2.9). The other equalities are obtained by taking partial derivatives of (2.2.9) □

With this lemma, we have described the distribution of $j^2_x(f)$ only in terms of E. Since $j^0_x(f)$ and $j^1_x(f)$ are the projections of $j^2_x(f)$ onto $\mathbb{R}^r$ and $\mathbb{R}^r\otimes(\mathbb{R}\oplus T^*_xM)$ respectively, their distributions are also characterized by Lemma 2.2.4.

2.2.5 Riemannian random waves

In this section, we describe what we called Riemannian random waves, that is random linear combinations of eigenfunctions of the Laplacian.

Let Δ denote the Laplace-Beltrami operator on the closed Riemannian manifold (M,g). Recall the following classical facts from the theory of elliptical operators, see [GHL04, thm. 4.43].

Theorem 2.2.5. 1. *The eigenvalues of $\Delta : \mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ can be arranged into an increasing sequence of non-negative numbers $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \xrightarrow[k\to+\infty]{} +\infty$.*

2. *The associated eigenspaces are finite-dimensional, and they are pairwise orthogonal for the L^2-inner product* (2.1.1) *on $\mathcal{C}^\infty(M)$ induced by g.*

Let $\lambda \geqslant 0$, then we denote by V_λ the subspace of $\mathcal{C}^\infty(M)$ spanned by the eigenfunctions of Δ associated to eigenvalues that are less or equal to λ. Each $f = (f^{(1)},\dots,f^{(r)}) \in (V_\lambda)^r$ defines naturally a map from M to $\mathbb{R}^r$ so that we can see $(V_\lambda)^r$ as a subspace of $\mathcal{C}^\infty(M,\mathbb{R}^r)$. By Theorem 2.2.5, V_λ is finite-dimensional so we can apply the construction of sections 2.2.1 to 2.2.4 to $(V_\lambda)^r$.

Since we consider a product situation, it is possible to make several simplifications. First, the scalar product on $(V_\lambda)^r$ is induced by the one on V_λ. Thus $f = (f^{(1)},\dots,f^{(r)})$ is a standard Gaussian in $(V_\lambda)^r$ if and only if $f^{(1)},\dots,f^{(r)}$ are r independent standard Gaussian in V_λ. For a $f\in (V_\lambda)^r$ satisfying this condition, $f^{(1)},\dots,f^{(r)}$ are independent, and so are their derivatives of any order. This means that, for every $x\in M$, the matrices of $\operatorname{Var}(f(x))$, $\operatorname{Var}(()\, d_xf)$ and $\operatorname{Var}\big(\nabla^2_x f\big)$ in the canonical basis of $\mathbb{R}^r$ are block diagonal.

Another way to say this is that the kernel of $(V_\lambda)^r$ is a product in the following sense. We denote by $e_\lambda : M\times M\to\mathbb{R}$ the Schwartz kernel of the orthogonal projection onto V_λ in $\mathcal{C}^\infty(M)$ and by E_λ the Schwartz kernel of $(V_\lambda)^r$. The kernel e_λ is the *spectral function of the Laplacian* and precise asymptotics are known for e_λ and its derivatives, see section 2.3.

Let $(\varphi_1,\dots,\varphi_N)$ be an orthonormal basis of V_λ. By (2.2.7),

$$e_\lambda : (x,y) \mapsto \sum_{i=1}^N \varphi_i(x)\varphi_i(y). \tag{2.2.15}$$

Let $(\zeta_1, \dots, \zeta_r)$ denote the canonical basis of $\mathbb{R}^r$. The maps $\varphi_i \zeta_q : M \to \mathbb{R}^r$ with $1 \leqslant i \leqslant N$ and $1 \leqslant q \leqslant r$ give an orthonormal basis of $(V_\lambda)^r$ and, for all x and $y \in M$,

$$E_\lambda(x, y) = \sum_{q=1}^{r} \sum_{i=1}^{N} (\varphi_i(x)\zeta_q) \otimes (\varphi_i(y)\zeta_q) = e_\lambda(x, y) \sum_{q=1}^{r} \zeta_q \otimes \zeta_q.$$

Lemma 2.2.6. *Let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{R}^r$, for all x, $y \in M$,*

$$E_\lambda(x, y) = e_\lambda(x, y) \left(\sum_{q=1}^{r} \zeta_q \otimes \zeta_q \right).$$

An immediate consequence of this and Lemma 2.2.2 is that $(V_\lambda)^r$ is 0-ample if and only if $e_\lambda(x, x) > 0$ for all $x \in M$. By (2.2.15), this is equivalent to V_λ being base-point-free, that is for every $x \in M$, there exists $f \in V_\lambda$ such that $f(x) \neq 0$.

Lemma 2.2.7. *For all $\lambda \geqslant 0$, $(V_\lambda)^r$ is 0-ample.*

Proof. The constant functions on M are eigenfunctions of Δ associated to the eigenvalue 0. Thus for every $\lambda \geqslant 0$, V_λ contains all constant functions on M, hence is base-point-free. By the above remark, $(V_\lambda)^r$ is then 0-ample. □

2.2.6 The real algebraic setting

Let us now describe more precisely the real algebraic framework. The main difference with what we did previously is that we consider sections of a rank r vector bundle instead of maps to $\mathbb{R}^r$. The local picture is the same as in sections 2.2.1 to 2.2.5, so that we can adapt everything to this setting. But the formalism is a bit heavier. A classical reference for most of this material is [GH94].

Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension n. We equip $\mathcal{X}$ with a real structure, that is with an antiholomorphic involution $c_\mathcal{X}$. We assume that its real locus, the set of fixed points of $c_\mathcal{X}$, is not empty and we denote it by $\mathbb{R}\mathcal{X}$. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ equipped with a real structure $c_\mathcal{L}$ compatible with the one on $\mathcal{X}$. By this we mean that $c_\mathcal{X} \circ \pi = \pi \circ c_\mathcal{L}$, where $\pi : \mathcal{L} \to \mathcal{X}$ stands for the projection map onto the base space. Similarly, let $\mathcal{E}$ be a holomorphic vector bundle of rank r over $\mathcal{X}$, with a compatible real structure $c_\mathcal{E}$.

Let $h_\mathcal{L}$ and $h_\mathcal{E}$ be real Hermitian metrics on $\mathcal{L}$ and $\mathcal{E}$ respectively, that is $c_\mathcal{L}^*(h_\mathcal{L}) = \overline{h_\mathcal{L}}$ and $c_\mathcal{E}^*(h_\mathcal{E}) = \overline{h_\mathcal{E}}$. We assume that $h_\mathcal{L}$ is positive in the sense that its *curvature form* ω is Kähler. Locally we have:

$$\omega_{/\Omega} = \frac{1}{2i} \partial\bar{\partial} \ln \left(h_\mathcal{L}(\zeta, \zeta) \right)$$

where ζ is any non-vanishing local holomorphic section of $\mathcal{L}$ on the open set $\Omega \subset \mathcal{X}$. This form corresponds to a Hermitian metric $g_\mathbb{C} = \omega(\cdot, i\cdot)$ on $\mathcal{X}$ whose real part is a Riemannian metric g. We denote by $\mathrm{d}V_\mathcal{X}$ the volume form $\frac{\omega^n}{n!}$ on $\mathcal{X}$.

Remark 2.2.8. The normalization of ω is the one of [BBS08, BSZ00], but differs from our references concerning peak sections [GW15a, Tia90]. This will cause some discrepancies with the latter two in section 2.3.2. With our convention, the Fubini-Study metric on $\mathbb{RP}^n$ induced by the standard metric on the hyperplane line bundle $\mathcal{O}(1)$ is the quotient of the Euclidean metric on $\mathbb{S}^n$.

Let $d \in \mathbb{N}$, then the vector bundle $\mathcal{E} \otimes \mathcal{L}^d$ comes with a real structure $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d$ compatible with $c_{\mathcal{X}}$ and a real Hermitian metric $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$. We equip the space $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ with the L^2 Hermitian product defined by:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \qquad \langle s_1 \, , s_2 \rangle = \int_{\mathcal{X}} h_d(s_1, s_2) \, \mathrm{d}V_{\mathcal{X}}. \tag{2.2.16}$$

We know from the vanishing theorem of Kodaira and Serre that the space $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$ has finite dimension N_d and that N_d grows as a polynomial of degree n in d, when d goes to infinity. We denote by:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0\left(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d\right) \;\middle|\; c_d \circ s = s \circ c_{\mathcal{X}} \right\}$$

the space of real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$, which has real dimension N_d. The Hermitian product (2.2.16) induces a Euclidean inner product on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Notice that we integrate on the whole of $\mathcal{X}$, not only on the real locus, even when we consider real sections.

If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is such that its restriction to $\mathbb{R}\mathcal{X}$ vanishes transversally, then its real zero set $Z_s = s^{-1}(0) \cap \mathbb{R}\mathcal{X}$ is a (possibly empty) submanifold of $\mathbb{R}\mathcal{X}$ of codimension r. We denote by $|\mathrm{d}V_s|$ the Riemannian measure induced on this submanifold by the metric g.

As in section 2.2.2, we consider the incidence manifold:

$$\Sigma_d = \left\{ (s, x) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \;\middle|\; s(x) = 0 \right\}. \tag{2.2.17}$$

In this setting, Σ_d is the zero set of the bundle map $F_d : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \to \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathbb{R}\mathcal{X}$ defined by $F_d : (s, x) \mapsto s(x)$. In a trivialization, the situation is similar to the one in section 2.2.2. Thus, if $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample, Σ_d is a smooth manifold equipped with two projection maps, π_1 and π_2, onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and $\mathbb{R}\mathcal{X}$ respectively. By Sard's theorem, the discriminant locus of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then has measure 0 for any non-singular Gaussian measure, since it is the set of critical values of π_1.

Remark 2.2.9. Here, by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample we mean that, for every $x \in \mathbb{R}\mathcal{X}$, the evaluation map $j_x^{0,d} : s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \mapsto s(x) \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ is onto.

Let ∇^d denote any real connection on $\mathcal{E} \otimes \mathcal{L}^d$, that is such that for every smooth section s, $\nabla^d (c_d \circ s \circ c_{\mathcal{X}}) = c_d \circ (\nabla^d s) \circ dc_{\mathcal{X}}$. For example one could choose the Chern connection. We consider the vertical component $\nabla^d F_d$ of the differential of F_d, whose kernel is the tangent space of Σ_d. For any $(s_0, x) \in \Sigma_d$ the partial derivatives of F_d are given by:

$$\partial_1^d F_d(s_0, x) = j_x^{0,d} \qquad \text{and} \qquad \partial_2^d F_d(s_0, x) = \nabla_x^d s_0. \tag{2.2.18}$$

Note that we only consider points of the zero section of $\mathcal{E} \otimes \mathcal{L}^d$, hence all this does not depend on the choice of ∇^d.

Let P_1 (resp. P_2) denote the projection from $\mathcal{X} \times \mathcal{X}$ onto the first (resp. second) factor. Recall that $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ stands for the bundle $P_1^*(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \otimes P_2^*(\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathcal{X} \times \mathcal{X}$. Let E_d denote the Schwartz kernel of the orthogonal projection from the space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. It is the unique section of $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ such that, for every real smooth section s of $\mathcal{E} \otimes \mathcal{L}^d$, the projection of s on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is given by:

$$x \mapsto \langle E_d(x, \cdot) \, , s \rangle = \int_{y \in \mathcal{X}} h_d(E_d(x, y), s(y)) \, \mathrm{d}V_{\mathcal{X}}.$$

Here, h_d acts on the second factor of $(\overline{\mathcal{E} \otimes \mathcal{L}^d})_x \otimes (\mathcal{E} \otimes \mathcal{L}^d)_y$.

The kernel E_d satisfies a reproducing kernel property similar to (2.2.6):

$$\forall s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d),\ \forall x \in \mathbb{R}\mathcal{X}, \qquad s(x) = \langle E_d(x, \cdot)\,, s\rangle \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x. \tag{2.2.19}$$

If $(s_1, \dots, s_{N_d})$ is an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then for all x and $y \in \mathcal{X}$,

$$E_d(x, y) = \sum_{i=1}^{N_d} s_i(x) \otimes s_i(y). \tag{2.2.20}$$

For any $x \in \mathbb{R}\mathcal{X}$, this shows $E_d(x, x)$ is in the span $\{\zeta \otimes \zeta \mid \zeta \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x\}$. We also get the analogue of Lemma 2.2.7.

Lemma 2.2.10. *$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample if and only if for any $x \in \mathbb{R}\mathcal{X}$, $E_d(x, x)$ is a positive-definite bilinear form on $(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)^*$.*

Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, then $(s(x))_{x\in\mathcal{X}}$ defines a centered Gaussian field with values in $\mathcal{E} \otimes \mathcal{L}^d$ and its covariance function is a section of $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$. The same proof as for Proposition 2.2.3 gives the following.

Proposition 2.2.11. *Let E_d be the Schwartz kernel of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let $s \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, then we have:*

$$\forall x, y \in \mathcal{X}, \qquad \mathrm{Cov}(s(x), s(y)) = \mathbb{E}[s(x) \otimes s(y)] = E_d(x, y).$$

Remark 2.2.12. The kernel E_d is also the kernel of the orthogonal projection onto $H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ in $\Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ for the Hermitian inner product (2.2.16), that is the *Bergman kernel* of $\mathcal{E} \otimes \mathcal{L}^d$.

As in section 2.2.4, let $\nabla^{\mathbb{R}\mathcal{X}}$ denote the Levi-Civita connection on $(\mathbb{R}\mathcal{X}, g)$. This connection and ∇^d induce the connection $\nabla^{\mathbb{R}\mathcal{X}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^d$ on $T^*(\mathbb{R}\mathcal{X}) \otimes (\mathcal{E} \otimes \mathcal{L}^d)$. We denote by $\nabla^{2,d}$ the second covariant derivative $(\nabla^{\mathbb{R}\mathcal{X}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^d) \circ \nabla^d$.

Let $x \in \mathbb{R}\mathcal{X}$ and let $\mathcal{J}^k_x(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of real k-jets of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ at x. On the space of smooth real sections of $\mathcal{E} \otimes \mathcal{L}^d$ we define

$$j^{1,d}_x : s \mapsto (s(x), \nabla^d_x s) \qquad \text{and} \qquad j^{2,d}_x : s \mapsto (s(x), \nabla^d_x s, \nabla^{2,d}_x s),$$

where $\nabla^d_x s$ and $\nabla^{2,d}_x s$ are implicitly restricted to $T_x\mathbb{R}\mathcal{X}$. These maps induce isomorphisms from $\mathcal{J}^1_x(\mathcal{E} \otimes \mathcal{L}^d)$ to the image of $j^{1,d}_x$ in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathbb{R} \oplus T^*_x\mathbb{R}\mathcal{X})$ and from $\mathcal{J}^2_x(\mathcal{E} \otimes \mathcal{L}^d)$ to the image of $j^{2,d}_x$ in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathbb{R} \oplus T^*_x\mathbb{R}\mathcal{X} \oplus \mathrm{Sym}(T^*_x\mathbb{R}\mathcal{X}))$ respectively. Note that the above isomorphisms are not canonical since they depend on the choice of ∇^d.

We have the following equivalent of Lemma 2.2.4, with the same proof, and similar notations for the partial covariant derivatives.

Lemma 2.2.13. *Let E_d denote the Bergman kernel of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let $x \in \mathbb{R}\mathcal{X}$, then $j^{2,d}_x(s)$ is a centered Gaussian vector, and its variance form is characterized by:*

$$\mathrm{Var}(s(x)) = \mathbb{E}[s(x) \otimes s(x)] = E_d(x, x), \tag{2.2.21}$$

$$\mathrm{Var}\left(\nabla^d_x s\right) = \mathbb{E}\left[\nabla^d_x s \otimes \nabla^d_x s\right] = (\partial_x\partial_y E_d)(x, x), \tag{2.2.22}$$

$$\mathrm{Var}\left(\nabla^{2,d}_x s\right) = \mathbb{E}\left[\nabla^{2,d}_x s \otimes \nabla^{2,d}_x s\right] = (\partial_{x,x}\partial_{y,y} E_d)(x, x), \tag{2.2.23}$$

$$\mathrm{Cov}\left(s(x), \nabla^d_x s\right) = \mathbb{E}\left[s(x) \otimes \nabla^d_x s\right] = (\partial_y E_d)(x, x), \tag{2.2.24}$$

$$\mathrm{Cov}\left(s(x), \nabla^{2,d}_x s\right) = \mathbb{E}\left[s(x) \otimes \nabla^{2,d}_x s\right] = (\partial_{y,y} E_d)(x, x), \tag{2.2.25}$$

$$\mathrm{Cov}\left(\nabla^d_x s, \nabla^{2,d}_x s\right) = \mathbb{E}\left[\nabla^d_x s \otimes \nabla^{2,d}_x s\right] = (\partial_x\partial_{y,y} E_d)(x, x). \tag{2.2.26}$$

2.3 Estimates for the covariance kernels

We state in this section the estimates for the kernels described above and their first and second derivatives. These estimates will allow us to compute the limit distribution for the random 2-jets induced by the Gaussian field $(f(x))_{x\in M}$ (resp. $(s(x))_{x\in\mathbb{R}\mathcal{X}}$).

In the case of the spectral function of the Laplacian e_λ, the asymptotics of section 2.3.1 were established by Bin [Bin04], extending results of Hörmander [Hö68]. In the algebraic case, Bleher, Shiffman and Zelditch used estimates for the related Szegö kernel, see [BSZ00, thm. 3.1]. In terms of the Bergman kernel, a similar result was established in [BBS08]. Both these results concern line bundles. Here, we establish the estimates we need for the Bergman kernel in the case of a higher rank bundle using Hörmander–Tian peak sections (see sections 2.3.2 and 2.3.3 below).

2.3.1 The spectral function of the Laplacian

We consider the Riemannian setting of section 2.2.5. Let $x \in M$ and let $(x_1, \dots, x_n)$ be normal coordinates centered at x. Let $(y_1, \dots, y_n)$ denote the same coordinates in a second copy of M, so that $(x_1, \dots, x_n, y_1, \dots, y_n)$ are normal coordinates around $(x, x) \in M \times M$. We denote by ∂_{x_i} (resp. ∂_{y_i}) the partial derivative with respect to x_i (resp. y_i), and similarly ∂_{x_i,x_j} (resp. ∂_{y_i,y_j}) denotes the second derivative with respect to x_i and x_j (resp. y_i and y_j). Let

$$\gamma_0 = \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2}\right)}, \qquad \gamma_1 = \frac{1}{2(4\pi)^{\frac{n}{2}}\Gamma\left(2+\frac{n}{2}\right)} \qquad \text{and} \qquad \gamma_2 = \frac{1}{4(4\pi)^{\frac{n}{2}}\Gamma\left(3+\frac{n}{2}\right)},$$

where Γ is Euler's gamma function. Let us recall the main theorem of [Bin04].

Theorem 2.3.1 (Bin)**.** *Let V_λ be as in section 2.2.5 and let e_λ denote its Schwartz kernel. The following asymptotics hold, uniformly in $x \in M$, as $\lambda \to +\infty$:*

$$e_\lambda(x,x) = \gamma_0 \lambda^{\frac{n}{2}} + O\left(\lambda^{\frac{n-1}{2}}\right), \tag{2.3.1}$$

$$\partial_{x_i} e_\lambda(x,x) = O\left(\lambda^{\frac{n}{2}}\right), \tag{2.3.2}$$

$$\partial_{x_i,x_k} e_\lambda(x,x) = \begin{cases} -\gamma_1 \lambda^{\frac{n}{2}+1} + O\left(\lambda^{\frac{n+1}{2}}\right) & \text{if } i = k, \\ O\left(\lambda^{\frac{n+1}{2}}\right) & \text{if } i \neq k, \end{cases} \tag{2.3.3}$$

$$\partial_{x_i}\partial_{y_j} e_\lambda(x,x) = \begin{cases} \gamma_1 \lambda^{\frac{n}{2}+1} + O\left(\lambda^{\frac{n+1}{2}}\right) & \text{if } i = j, \\ O\left(\lambda^{\frac{n+1}{2}}\right) & \text{if } i \neq j, \end{cases} \tag{2.3.4}$$

$$\partial_{x_i,x_k}\partial_{y_j} e_\lambda(x,x) = O\left(\lambda^{\frac{n}{2}+1}\right), \tag{2.3.5}$$

$$\partial_{x_i,x_k}\partial_{y_j,y_l} e_\lambda(x,x) = \begin{cases} 3\gamma_2 \lambda^{\frac{n}{2}+2} + O\left(\lambda^{\frac{n+3}{2}}\right) & \text{if } i = j = k = l, \\ \gamma_2 \lambda^{\frac{n}{2}+2} + O\left(\lambda^{\frac{n+3}{2}}\right) & \text{if } i = j \neq k = l \text{ or } i = l \neq j = k \\ & \text{or } i = k \neq j = l, \\ O\left(\lambda^{\frac{n+3}{2}}\right) & \text{otherwise.} \end{cases} \tag{2.3.6}$$

Since e_λ is symmetric, this also gives the asymptotics for $\partial_{y_j} e_\lambda$, $\partial_{y_j,y_l} e_\lambda$ and $\partial_{x_i}\partial_{y_j,y_l} e_\lambda$ along the diagonal. This theorem, together with Lemma 2.2.6, gives the estimates we need for the kernel E_λ of $(V_\lambda)^r$. We will need the following relations:

$$\frac{\gamma_0}{\gamma_1} = n+2 \qquad \text{and} \qquad \frac{\gamma_1}{\gamma_2} = n+4. \tag{2.3.7}$$

2.3.2 Hörmander–Tian peak sections

We now recall the construction of Hörmander–Tian peak sections in the framework of section 2.2.6. Let $\mathcal{X}$ be a complex projective manifold. Let $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle and $\mathcal{L}$ be an ample holomorphic Hermitian line bundle, both defined over $\mathcal{X}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures, and that the Kähler metric $g_{\mathbb{C}}$ on $\mathcal{X}$ is induced by the curvature ω of $\mathcal{L}$.

The goal of this subsection is to build, for every d large enough, a family of real sections of $\mathcal{E}\otimes\mathcal{L}^d$ with prescribed 2-jets at some fixed point $x \in \mathbb{R}\mathcal{X}$. Moreover, we want this family to be orthonormal, up to an error that goes to 0 as d goes to infinity. Using these sections, we will compute (in section 2.3.3 below) the asymptotics we need for the Bergman kernel.

Let $x \in \mathbb{R}\mathcal{X}$ and $(x_1, \dots, x_n)$ be real holomorphic coordinates centered at x and such that $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is orthonormal at x. The next lemma is established in [GW14b, lemma 3.3], up to a factor π coming from different normalizations of the metric.

Lemma 2.3.2. *There exists a real holomorphic frame ζ_0 for $\mathcal{L}$, defined over some neighborhood of x, whose potential $-\ln(h_{\mathcal{L}}(\zeta_0,\zeta_0))$ vanishes at x, where it reaches a local minimum with Hessian $g_{\mathbb{C}}$.*

We choose such a frame ζ_0. Let $(\zeta_1, \dots, \zeta_r)$ be a real holomorphic frame for $\mathcal{E}$ over a neighborhood of x, which is orthonormal at x. Since $\mathcal{X}$ is compact, we can find $\rho > 0$, not depending on x, such that local coordinates and frames as above are defined at least on the geodesic ball of radius ρ centered at x. The following results are proved in [GW15a, section 2.3]. See also [GW16, section 2.2] and the paper by Tian [Tia90, lemmas 1.2 and 2.3], without the higher rank bundle $\mathcal{E}$ but with more details.

Proposition 2.3.3. *Let $p = (p_1, \dots, p_n) \in \mathbb{N}^n$, $m > p_1 + \dots + p_n$ and $q \in \{1, \dots, r\}$. There exists $d_0 \in \mathbb{N}$ such that, for any $d \geqslant d_0$, there exist $C_{d,p} > 0$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ such that $\|s\| = 1$ and*

$$s(x_1, \dots, x_n) = C_{d,p} \left(x_1^{p_1} \cdots x_n^{p_n} + O\Big(\|(x_1, \dots, x_n)\|^{2m}\Big)\right) \left(1 + O\big(d^{-2m}\big)\right) \zeta_q \otimes \zeta_0^d$$

in some neighborhood of x, where the estimate $O(d^{-2m})$ is uniform in $x \in \mathbb{R}\mathcal{X}$.

Moreover, d_0 depends on m but does not depend on x, p, q or our choices of local coordinates and frames. Finally, $C_{d,p}$ is given by:

$$(C_{d,p})^{-2} = \int_{\left\{\|(x_1,\dots,x_n)\| \leqslant \frac{\ln(d)}{\sqrt{d}}\right\}} |x_1^{p_1} \cdots x_n^{p_n}|^2 \, h_{\mathcal{L}}^d \left(\zeta_0^d, \zeta_0^d\right) \mathrm{d}V_{\mathcal{X}}.$$

The following definitions use Proposition 2.3.3 with $m = 3$ and the corresponding d_0.

Definitions 2.3.4. For any $d \geqslant d_0$ and $q \in \{1, \dots, r\}$, the sections of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ given by Proposition 2.3.3 are denoted by:

- $s_0^{d,q}$ for $p_1 = \dots = p_n = 0$,
- $s_i^{d,q}$ for $p_i = 1$ and $\forall\, k \neq i$, $p_k = 0$,
- $s_{i,i}^{d,q}$ for $p_i = 2$ and $\forall\, k \neq i$, $p_k = 0$,
- $s_{i,j}^{d,q}$ for $p_i = p_j = 1$ and $\forall\, k \notin \{i, j\}$, $p_k = 0$, when $i < j$.

Computing the values of the corresponding $C_{d,p}$ (see [GW16, lemma 2.5]), we get the following asymptotics as d goes to infinity. Once again, $O(d^{-1})$ is uniform in x.

Lemma 2.3.5. *For every $q \in \{1, \dots, r\}$, we have:*

$$s_0^{d,q} = \sqrt{\frac{d^n}{\pi^n}} \left(1 + O\Big(\|(x_1, \dots, x_n)\|^6\Big)\right) \left(1 + O\big(d^{-1}\big)\right) \zeta_q \otimes \zeta_0^d, \quad (2.3.8)$$

$$\forall i \in \{1, \dots, n\}, \quad s_i^{d,q} = \sqrt{\frac{d^{n+1}}{\pi^n}} \left(x_i + O\Big(\|(x_1, \dots, x_n)\|^6\Big)\right) \left(1 + O\big(d^{-1}\big)\right) \zeta_q \otimes \zeta_0^d, \quad (2.3.9)$$

$$\forall i \in \{1, \dots, n\}, \quad s_{i,i}^{d,q} = \sqrt{\frac{d^{n+2}}{\pi^n}} \left(\frac{x_i^2}{\sqrt{2}} + O\Big(\|(x_1, \dots, x_n)\|^6\Big)\right) \left(1 + O\big(d^{-1}\big)\right) \zeta_q \otimes \zeta_0^d, \quad (2.3.10)$$

and finally, $\forall i, j \in \{1, \dots, n\}$ such that $i < j$,

$$s_{i,j}^{d,q} = \sqrt{\frac{d^{n+2}}{\pi^n}} \left(x_i x_j + O\Big(\|(x_1, \dots, x_n)\|^6\Big)\right) \left(1 + O\big(d^{-1}\big)\right) \zeta_q \otimes \zeta_0^d. \quad (2.3.11)$$

Remark 2.3.6. These values differ from the one given in [GW15a, lemma 2.3.5] by a factor $\left(\int_{\mathcal{X}} \mathrm{d}V_{\mathcal{X}}\right)^{\frac{1}{2}}$ and some power of $\sqrt{\pi}$ because we do not use the same normalization for the volume form. For the same reason they also differ from [Tia90, lemma 2.3] by a factor $\pi^{\frac{n}{2}}$.

The sections defined in Definition 2.3.4 are linearly independent, at least for d large enough. In fact, they are asymptotically orthonormal in the following sense. Let us denote by $H_{2,x}^d \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the subspace of sections that vanish up to order 2 at x.

Lemma 2.3.7. *The sections $(s_i^{d,q})_{\substack{1 \leqslant q \leqslant r \\ 0 \leqslant i \leqslant n}}$ and $(s_{i,j}^{d,q})_{\substack{1 \leqslant q \leqslant r \\ 1 \leqslant i \leqslant j \leqslant n}}$ defined in Definition 2.3.4 have L^2-norm equal to 1 and their pairwise scalar product is dominated by a $O(d^{-1})$ independent of x. Moreover, their scalar product with any unit element of $H_{2,x}^d$ is dominated by some $O(d^{-1})$ not depending on x.*

2.3.3 The Bergman kernel

In this section we compute some asymptotics for the Bergman kernel and its derivatives. Let $x \in \mathbb{R}\mathcal{X}$ and let $(x_1, \dots, x_n)$ be real holomorphic coordinates around x such that $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is orthonormal at x. We denote by $(y_1, \dots, y_n)$ the same coordinates as $(x_1, \dots, x_n)$ in a second copy of $\mathcal{X}$. Let ζ_0 be a real holomorphic frame for $\mathcal{L}$ given by Lemma 2.3.2 and $(\zeta_1, \dots, \zeta_r)$ be a real holomorphic frame for $\mathcal{E}$ that is orthonormal at x. For simplicity, we set $\zeta_p^d = \zeta_p(x) \otimes \zeta_0^d(x)$ for every $p \in \{1, \dots, r\}$ and $d \in \mathbb{N}$, so that $(\zeta_1^d, \dots, \zeta_r^d)$ is an orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$.

Let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$ such that, for every $p \in \{1, \dots, r\}$, $\nabla^d(\zeta_p \otimes \zeta_0^d)$ vanishes in a neighborhood of x. In this neighborhood, we have for every function f:

$$\nabla^d(f\zeta_p \otimes \zeta_0^d) = df \otimes \zeta_p \otimes \zeta_0^d \qquad \text{and} \qquad \nabla^{2,d}(f\zeta_p \otimes \zeta_0^d) = \nabla^2 f \otimes \zeta_p \otimes \zeta_0^d, \quad (2.3.12)$$

where $\nabla^{2,d}$ stands for the associated second covariant derivative, as in section 2.2.6. This choice of connection may seem restrictive but the quantity we want to compute do not depend on a choice of connection so that we can choose one that suits us.

As usual, ∇^d induces a connection on $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$. We denote by $\partial_{x_i}^d$ and $\partial_{y_i}^d$ the partial covariant derivatives with respect to x_i and y_i respectively. We also denote by ∂_{x_i,x_j}^d (resp. ∂_{y_i,y_j}^d) the second derivative with respect to x_i and x_j (resp. y_i and y_j).

Proposition 2.3.8. *The following asymptotics hold as $d \to +\infty$. They are independent of $x \in \mathbb{R}\mathcal{X}$ and of the choice of the holomorphic frame $(\zeta_1, \dots, \zeta_r)$.*

$$\left\langle E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = \begin{cases} \dfrac{d^n}{\pi^n} + O\big(d^{n-1}\big) & \text{if } p = p', \\ O\big(d^{n-1}\big) & \text{otherwise,} \end{cases} \tag{2.3.13}$$

$$\left\langle \partial_{x_i}^d E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = O\Big(d^{n-\frac{1}{2}}\Big)\,, \tag{2.3.14}$$

$$\left\langle \partial_{x_i,x_k}^d E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = O(d^n)\,, \tag{2.3.15}$$

$$\left\langle \partial_{x_i}^d \partial_{y_j}^d E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = \begin{cases} \dfrac{d^{n+1}}{\pi^n} + O(d^n) & \text{if } p = p' \text{ and } i = j, \\ O(d^n) & \text{otherwise,} \end{cases} \tag{2.3.16}$$

$$\left\langle \partial_{x_i,x_k}^d \partial_{y_j}^d E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = O\Big(d^{n+\frac{1}{2}}\Big)\,, \tag{2.3.17}$$

$$\left\langle \partial_{x_i,x_k}^d \partial_{y_j,y_l}^d E_d(x,x)\,, \zeta_p^d \otimes \zeta_{p'}^d \right\rangle = \begin{cases} 2\dfrac{d^{n+2}}{\pi^n} + O\big(d^{n+1}\big) & \text{if } p = p' \text{ and } i = j = k = l, \\ \dfrac{d^{n+2}}{\pi^n} + O\big(d^{n+1}\big) & \text{if } p = p' \text{ and } i = j \neq k = l, \\ & \text{or if } p = p' \text{ and } i = l \neq k = j, \\ O\big(d^{n+1}\big) & \text{otherwise.} \end{cases} \tag{2.3.18}$$

Remark 2.3.9. Since $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is not a product space, the terms with $p \neq p'$ are usually not zero. However they are zero when $\mathcal{E}$ is trivial, for example.

Corollary 2.3.10. *For every d large enough, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is* 0*-ample.*

Proof of Corollary 2.3.10. Let $x \in \mathbb{R}\mathcal{X}$. By (2.3.13), the matrix of $E_d(x,x)$ in any orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L})_x$ is $\left(\frac{d}{\pi}\right)^n I_r \left(1 + O\big(d^{-1}\big)\right)$, where I_r stands for the identity matrix of size r. Then $E_d(x,x)$ is positive-definite for d larger than some d_0 independent of x. By Lemma 2.2.10, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample for $d \geqslant d_0$. $\square$

Proof of Proposition 2.3.8. First we build an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ by applying the Gram–Schmidt process to the family of peak sections. Then we use formula (2.2.20) and the asymptotics of Lemma 2.3.5 to prove the proposition.

We order the sections of Definition 2.3.4 as follows:

$$\begin{aligned} &s_0^{d,1}, \dots, s_0^{d,r}, s_1^{d,1}, \dots, s_1^{d,r}, \dots, s_n^{d,1}, \dots, s_n^{d,r}, s_{1,1}^{d,1}, \dots, s_{1,1}^{d,r}, s_{2,2}^{d,1}, \dots, s_{2,2}^{d,r}, \dots, s_{n,n}^{d,1}, \dots, s_{n,n}^{d,r}, \\ &s_{1,2}^{d,1}, \dots, s_{1,2}^{d,r}, s_{1,3}^{d,1}, \dots, s_{1,3}^{d,r}, \dots, s_{1,n}^{d,1}, \dots, s_{1,n}^{d,r}, s_{2,3}^{d,1}, \dots, s_{2,3}^{d,r}, \dots, s_{n-1,n}^{d,1}, \dots, s_{n-1,n}^{d,r}. \end{aligned} \tag{2.3.19}$$

This family is linearly independent for d large enough and spans a space whose direct sum with $H_{2,x}^d$ is $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. We complete it into a basis B of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ by adding an orthonormal basis of $H_{2,x}^d$ at the end of the previous list.

We apply the Gram–Schmidt process to B, starting by the last elements and going backwards. Let $\widetilde{B}$ denote the resulting orthonormal basis, and $\widetilde{s}_0^{d,1}, \dots, \widetilde{s}_n^{d,r}, \widetilde{s}_{1,1}^{d,1}, \dots, \widetilde{s}_{n-1,n}^{d,r}$ denote its first elements. This way, $\widetilde{s}_{n-1,n}^{d,r}$ is a linear combination of $s_{n-1,n}^{d,r}$ and elements of $H_{2,x}^d$, and $\widetilde{s}_0^{d,1}$ is a linear combination of (possibly) all elements of B. We denote by $(b_i)_{1 \leqslant i \leqslant \frac{r(n+1)(n+2)}{2}}$ the first elements of B listed above (2.3.19) and by $(\widetilde{b}_i)$ the corresponding elements of $\widetilde{B}$.

Let $i \in \left\{1, \ldots, \frac{r(n+1)(n+2)}{2}\right\}$ and assume that for any $k \leqslant i$ and any $j > i$ we have $\left\langle b_k \,, \widetilde{b}_j \right\rangle = O\big(d^{-1}\big)$. Note that this is the case for $i = \frac{r}{2}(n+1)(n+2)$. Then,

$$\widetilde{b}_i = \frac{b_i - \displaystyle\sum_{j>i} \left\langle b_i \,, \widetilde{b}_j \right\rangle \widetilde{b}_j - \pi_i}{\left\| b_i - \displaystyle\sum_{j>i} \left\langle b_i \,, \widetilde{b}_j \right\rangle \widetilde{b}_j - \pi_i \right\|} \tag{2.3.20}$$

where π_i stands for the projection of b_i onto $H^d_{2,x}$. By Lemma 2.3.7,

$$\|\pi_i\|^2 = \langle b_i \,, \pi_i \rangle = O\big(d^{-1}\big) \,.$$

Then by our hypothesis $\left\| b_i - \sum_{j>i} \left\langle b_i \,, \widetilde{b}_j \right\rangle \widetilde{b}_j - \pi_i \right\|^2 = 1 + O\big(d^{-1}\big)$. Using Lemma 2.3.7 and the above hypothesis once again, we get:

$$\left\langle b_k \,, \widetilde{b}_i \right\rangle = \big(1 + O\big(d^{-1}\big)\big) \left(\langle b_k \,, b_i \rangle - \sum_{j>i} \left\langle b_k \,, \widetilde{b}_j \right\rangle \left\langle b_i \,, \widetilde{b}_j \right\rangle - \langle b_k \,, \pi_i \rangle \right) = O\big(d^{-1}\big) \,,$$

for any $k < i$. By induction, for any $1 \leqslant i < j \leqslant \frac{r(n+1)(n+2)}{2}$, $\left\langle b_i \,, \widetilde{b}_j \right\rangle = O\big(d^{-1}\big)$. Then, using (2.3.20), for any $i \in \left\{1, \ldots, \frac{r(n+1)(n+2)}{2}\right\}$,

$$\widetilde{b}_i = \left(b_i + O\big(d^{-1}\big) \sum_{j>i} \widetilde{b}_j - \pi_i \right) \big(1 + O\big(d^{-1}\big)\big) \,.$$

Another induction gives:

$$\widetilde{b}_i = \left(b_i + O(d^{-1}) \sum_{j>i} b_j + \widetilde{\pi}_i \right) \big(1 + O\big(d^{-1}\big)\big) \,, \tag{2.3.21}$$

for any i, where $\widetilde{\pi}_i \in H^d_{2,x}$ is such that $\|\widetilde{\pi}_i\|^2 = O\big(d^{-1}\big)$. Moreover, all the estimates are independent of x and $i \in \left\{1, \ldots, \frac{r(n+1)(n+2)}{2}\right\}$.

Among the elements of $\widetilde{B}$, only $\widetilde{s}_0^{d,1}, \ldots, \widetilde{s}_0^{d,r}$ do not vanish at x. Using formula (2.2.20), we get $E_d(x,x) = \displaystyle\sum_{1 \leqslant q \leqslant r} \widetilde{s}_0^{d,q}(x) \otimes \widetilde{s}_0^{d,q}(x)$. Then,

$$\left\langle E_d(x,x) \,, \zeta^d_p \otimes \zeta^d_{p'} \right\rangle = \sum_{q=1}^{r} \left\langle \widetilde{s}_0^{d,q}(x) \,, \zeta^d_p \right\rangle \left\langle \widetilde{s}_0^{d,q}(x) \,, \zeta^d_{p'} \right\rangle .$$

Recall that $\widetilde{b}_0 = \widetilde{s}_0^{d,1}, \ldots, \widetilde{b}_r = \widetilde{s}_0^{d,r}$. Because of (2.3.21), for all $q \in \{1, \ldots, r\}$,

$$\begin{aligned} \left\langle \widetilde{s}_0^{d,q}(x) \,, \zeta^d_p \right\rangle &= \left\langle s_0^{d,q}(x) \,, \zeta^d_p \right\rangle + O\big(d^{-1}\big) \left(\sum_{q'=1}^{r} \left\langle s_0^{d,q'}(x) \,, \zeta^d_p \right\rangle \right) \\ &= \begin{cases} \left(\dfrac{d}{\pi}\right)^{\frac{n}{2}} \big(1 + O\big(d^{-1}\big)\big) & \text{if } p = q, \\ O\Big(d^{\frac{n}{2}-1}\Big) & \text{otherwise,} \end{cases} \end{aligned} \tag{2.3.22}$$

where the last equality comes from equation (2.3.8). This establishes (2.3.13).

Likewise,

$$\begin{aligned}\left\langle \partial^d_{x_i} E_d(x,x)\,,\zeta^d_p \otimes \zeta^d_{p'}\right\rangle &= \left\langle \sum_{1\leqslant q\leqslant r} \partial^d_{x_i} \widetilde{s}^{d,q}_0(x)\otimes \widetilde{s}^{d,q}_0(x)\,,\zeta^d_p\otimes\zeta^d_{p'}\right\rangle \\ &= \sum_{1\leqslant q\leqslant r} \left\langle \partial^d_{x_i}\widetilde{s}^{d,q}_0(x)\,,\zeta^d_p\right\rangle \left\langle \widetilde{s}^{d,q}_0(x)\,,\zeta^d_{p'}\right\rangle.\end{aligned}$$

The description (2.3.21) shows that $\partial^d_{x_i}\widetilde{s}^{d,q}_0(x)$ does not necessarily vanish, but it equals:

$$O\big(d^{-1}\big) \sum_{\substack{1\leqslant q'\leqslant r\\ 1\leqslant j\leqslant n}} \partial^d_{x_i} s^{d,q'}_j(x).$$

By (2.3.9), one gets that $\left\langle \partial^d_{x_i}\widetilde{s}^{d,q}_0(x)\,,\zeta^d_p\right\rangle = O\Big(d^{\frac{n-1}{2}}\Big)$, for all p and q. Besides by (2.3.22), $\left\langle \widetilde{s}^{d,q}_0(x)\,,\zeta^d_p\right\rangle = O\Big(d^{\frac{n}{2}}\Big)$ for all p and q. This proves (2.3.14).

The remaining estimates can be proved in the same way, using Lemma 2.3.5 and the fact that the estimates for corresponding elements of $\widetilde{B}$ and B are the same. □

2.4 An integral formula for the Euler characteristic of a submanifold

The goal of this section is to derive an integral formula for the Euler characteristic of a submanifold defined as the zero set of some $f : M \to \mathbb{R}^r$, in terms of f and its derivatives. This section is independent of the previous ones and the results it contains are only useful for computing expected Euler characteristics (Theorems 2.1.2 and 2.1.4).

We start by recalling the formalism of double forms, which was already used in this context by Taylor and Adler, see [TA07, section 7.2]. The Riemann curvature tensor and the second fundamental form of a submanifold being naturally double forms, this provides a useful way to formulate the Chern–Gauss–Bonnet theorem and the Gauss equation. This is done in sections 2.4.2 and 2.4.3 respectively. Finally, we express the second fundamental form of a submanifold in terms of the derivatives of a defining function and prove the desired integral formula in section 2.4.4.

2.4.1 The algebra of double forms

We follow the exposition of [TA07, pp. 157–158]. Let V be a real vector space of dimension n. For p and $q \in \{0,\dots,n\}$ we denote by $\bigwedge^{p,q}(V^*)$ the space $\bigwedge^p(V^*)\otimes\bigwedge^q(V^*)$ of $(p+q)$-linear forms on V that are skew-symmetric in the first p and in the last q variables. The *space of double forms* on V is:

$$\textstyle\bigwedge^\bullet(V^*)\otimes\bigwedge^\bullet(V^*) = \displaystyle\bigoplus_{0\leqslant p,q\leqslant n} \textstyle\bigwedge^{p,q}(V^*). \tag{2.4.1}$$

Elements of $\bigwedge^{p,q}(V^*)$ are called (p,q)-*double forms*, or *double forms of type* (p,q). We set:

$$\textstyle\bigwedge^{\bullet,\bullet}(V^*) = \displaystyle\bigoplus_{p=0}^{n} \textstyle\bigwedge^{p,p}(V^*). \tag{2.4.2}$$

Note that $\bigwedge^{1,1} V^*$ is the space of bilinear forms on V.

On $\bigwedge^\bullet(V^*)\otimes\bigwedge^\bullet(V^*)$ we can define a double wedge product. It extends the usual wedge product on $\bigwedge^\bullet(V^*)\simeq\bigoplus_{p=0}^n\bigwedge^{p,0}(V^*)$, so we simply denote it by $\wedge$. For pure tensors $\alpha\otimes\beta$ and $\alpha'\otimes\beta'\in\bigwedge^\bullet(V^*)\otimes\bigwedge^\bullet(V^*)$, we set:

$$(\alpha\otimes\beta)\wedge(\alpha'\otimes\beta')=(\alpha\wedge\alpha')\otimes(\beta\wedge\beta') \tag{2.4.3}$$

and we extend $\wedge$ to all double forms by bilinearity. This makes $\bigwedge^\bullet(V^*)\otimes\bigwedge^\bullet(V^*)$ into an algebra, of which $\bigwedge^{\bullet,\bullet}(V^*)$ is a commutative subalgebra. We denote by $\gamma^{\wedge k}$ the double wedge product of a double form $\gamma\in\bigwedge^{\bullet,\bullet}(V^*)$ with itself k times.

Lemma 2.4.1. *Let α be a symmetric $(1,1)$-double form on V, then for every x,y,z and $w\in V$,*

$$\alpha^{\wedge 2}((x,y),(z,w))=2\left(\alpha(x,z)\alpha(y,w)-\alpha(x,w)\alpha(y,z)\right).$$

Proof. Let $(e_1,\dots,e_n)$ be a basis of V and $(e_1^*,\dots,e_n^*)$ its dual basis. We have:

$$\alpha=\sum_{1\leqslant i,k\leqslant n}\alpha_{ik}e_i^*\otimes e_k^* \qquad\text{and then}\qquad \alpha^{\wedge 2}=\sum_{1\leqslant i,j,k,l\leqslant n}\alpha_{ik}\alpha_{jl}(e_i^*\wedge e_j^*)\otimes(e_k^*\wedge e_l^*).$$

Note that we do not restrict ourselves to indices satisfying $i<j$ and $k<l$ as is usually the case with skew-symmetric forms. By multilinearity, it is sufficient to check the result on elements of the basis. Let i,j,k and $l\in\{1,\dots,n\}$, then

$$\begin{aligned}\alpha^{\wedge 2}((e_i,e_j),(e_k,e_l))&=\alpha_{ik}\alpha_{jl}-\alpha_{jk}\alpha_{il}-\alpha_{il}\alpha_{jk}+\alpha_{jl}\alpha_{ik}\\&=2(\alpha_{ik}\alpha_{jl}-\alpha_{il}\alpha_{jk}) \qquad\qquad\text{(since α is symmetric)}\\&=2(\alpha(e_i,e_k)\alpha(e_j,e_l)-\alpha(e_i,e_l)\alpha(e_j,e_k)).\end{aligned}$$ □

We can consider random vectors in spaces of double forms. The following technical result will be useful in the proofs of Theorems 2.1.2 and 2.1.4. See [TA07, lemma 12.3.1] for a proof.

Lemma 2.4.2. *Let V be a vector space of finite dimension n. Let α be a Gaussian vector in $\bigwedge^{1,1}V^*$. If α is centered, then for any $p\leqslant\frac{n}{2}$,*

$$\mathbb{E}\left[\alpha^{\wedge 2p}\right]=\frac{(2p)!}{2^p p!}\left(\mathbb{E}\left[\alpha^{\wedge 2}\right]\right)^{\wedge p}.$$

Assume now that V is endowed with an inner product. It induces a natural inner product on $\bigwedge^\bullet(V^*)$ such that, if $(e_1,\dots,e_n)$ is an orthonormal basis of V,

$$\left\{e_{i_1}^*\wedge\cdots\wedge e_{i_p}^*\;\middle|\;1\leqslant p\leqslant n\text{ and }1\leqslant i_1<i_2<\cdots<i_p\leqslant n\right\}$$

is an orthonormal basis of $\bigwedge^\bullet(V^*)$. We define the *trace operator* Tr on $\bigwedge^{\bullet,\bullet}(V^*)$ in the following way. If $\alpha\otimes\beta\in\bigwedge^{\bullet,\bullet}(V^*)$ is a pure tensor, then:

$$\mathrm{Tr}(\alpha\otimes\beta)=\langle\alpha\,,\beta\rangle \tag{2.4.4}$$

and we extend Tr to $\bigwedge^{\bullet,\bullet}(V^*)$ by linearity.

Let M be a smooth manifold of dimension n. Applying the previous construction pointwise to T_xM, we define the vector bundle $\bigwedge^\bullet(T^*M)\otimes\bigwedge^\bullet(T^*M)$ on M. Sections of this bundle are called *differential double forms* on M, and we can take the double wedge product of two such sections. Finally, if M is equipped with a Riemannian metric, we have a trace operator Tr which is defined pointwise by (2.4.4). This operator is $\mathcal{C}^\infty(M)$-linear and takes sections of the subbundle $\bigwedge^{\bullet,\bullet}(T^*M)=\bigoplus_{p=0}^n\bigwedge^{p,p}(T^*M)$ to smooth functions.

2.4.2 The Chern–Gauss–Bonnet theorem

Let (M, g) be a closed smooth Riemannian manifold of dimension n. We denote by ∇^M the Levi-Civita connection of M, and by κ its *curvature operator*. That is κ is the 2-form on M with values in the bundle $\mathrm{End}(TM) = TM \otimes T^*M$ defined by:

$$\kappa(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z$$

for any vector fields X, Y and Z. Here $[X,Y]$ is the Lie bracket of X and Y.

We denote by R the *Riemann curvature tensor* of M, defined by:

$$R(X,Y,Z,W) = g(\kappa(X,Y)W, Z),$$

for any vector fields X, Y, Z and W on M. This defines a four times covariant tensor on M which is skew-symmetric in the first two and in the last two variables, hence R can naturally be seen as a $(2,2)$-double form. All this is standard material, except for the very last point, see for example [Jos08, section 3.3].

We now state the Chern–Gauss–Bonnet theorem in terms of double forms. Recall that $|\mathrm{d}V_M|$ denotes the Riemannian measure on M (see (2.2.1)).

Theorem 2.4.3 (Chern–Gauss–Bonnet)**.** *Let M be a closed Riemannian manifold of even dimension n. Let R denote its Riemann curvature tensor and $\chi(M)$ denote its Euler characteristic. We have:*

$$\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}} \left(\frac{n}{2}\right)!} \int_M \mathrm{Tr}\left(R^{\wedge \frac{n}{2}}\right) |\mathrm{d}V_M| .$$

If M is orientable, this can be deduced from Atiyah-Singer's index theorem. The general case is treated in [Pal78]. The above formula in terms of double forms can be found in [TA07, thm. 12.6.18], up to a sign coming from different sign conventions in the definition of R.

Remark 2.4.4. If M is a closed manifold of odd dimension then $\chi(M) = 0$, see [Hat02, cor. 3.37].

2.4.3 The Gauss equation

Let (M, g) be a smooth Riemannian manifold of dimension n and $\widetilde{M}$ be a smooth submanifold of M of codimension $r \in \{1, \dots, n-1\}$. We denote by ∇^M and $\widetilde{\nabla}$ the Levi-Civita connections on M and $\widetilde{M}$ respectively. Likewise, we denote by R and $\widetilde{R}$ their Riemann curvature tensor. We wish to relate R and $\widetilde{R}$. This is done by the Gauss equation, see Proposition 2.4.5 below.

We denote by II the *second fundamental form* of $\widetilde{M} \subset M$ which is defined as the section of $T^\perp \widetilde{M} \otimes T^*\widetilde{M} \otimes T^*\widetilde{M}$ satisfying:

$$\mathrm{II}(X,Y) = -\left(\nabla^M_X Y - \widetilde{\nabla}_X Y\right) = -\left(\nabla^M_X Y\right)^\perp \tag{2.4.5}$$

for any vector fields X and Y on $\widetilde{M}$. Here, $\left(\nabla^M_X Y\right)^\perp$ stands for the orthogonal projection of $\nabla^M_X Y$ on $T^\perp \widetilde{M}$. It is well-known that II is symmetric in X and Y, see [Jos08, lemma 3.6.2].

The second fundamental form encodes the difference between $\widetilde{R}$ and R in the following sense, see [Jos08, thm. 3.6.2].

Proposition 2.4.5 (Gauss equation)**.** *Let X, Y, Z and W be vector fields on $\widetilde{M}$, then:*

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W) \rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle .$$

We want to write this Gauss equation in terms of double forms. Let $x \in \widetilde{M}$ and X, Y, Z and $W \in T_x\widetilde{M}$. Let $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM, by Lemma 2.A.8:

$$\begin{aligned}\langle \mathrm{II}(X,Z)\,, \mathrm{II}(Y,W)\rangle &- \langle \mathrm{II}(X,W)\,, \mathrm{II}(Y,Z)\rangle \\ &= \mathbb{E}[\langle \mathrm{II}(X,Z)\,, U\rangle \langle \mathrm{II}(Y,W)\,, U\rangle - \langle \mathrm{II}(X,W)\,, U\rangle \langle \mathrm{II}(Y,Z)\,, U\rangle].\end{aligned}$$

Then we apply Lemma 2.4.1 to the symmetric $(1,1)$-double form $\langle \mathrm{II}\,, U\rangle$ for fixed U. This gives:

$$\langle \mathrm{II}(X,Z)\,, \mathrm{II}(Y,W)\rangle - \langle \mathrm{II}(X,W)\,, \mathrm{II}(Y,Z)\rangle = \frac{1}{2}\,\mathbb{E}\Big[\langle \mathrm{II}\,, U\rangle^{\wedge 2}\left((X,Y),(Z,W)\right)\Big].$$

We proved the following version of the Gauss equation.

Proposition 2.4.6 (Gauss equation)**.** *Let (M,g) be a Riemannian manifold and let $\widetilde{M}$ be a smooth submanifold of M, such that $\dim(M) > \dim(\widetilde{M}) \geqslant 1$. Let R and $\widetilde{R}$ denote the Riemann curvature of M and $\widetilde{M}$ respectively, and let* II *be the second fundamental form of $\widetilde{M} \subset M$. Then, in the sense of double forms:*

$$\forall x \in \widetilde{M}, \qquad \widetilde{R}(x) = R(x) + \frac{1}{2}\,\mathbb{E}\Big[\langle \mathrm{II}(x)\,, U\rangle^{\wedge 2}\Big],$$

where $U \sim \mathcal{N}(0, \mathrm{Id})$ with values in T_xM, and $R(x)$ is implicitly restricted to $T_x\widetilde{M}$.

2.4.4 An expression for the second fundamental form

Let us now express the second fundamental form II of a submanifold $\widetilde{M}$ of M defined as the zero set of a smooth map $f : M \to \mathbb{R}^r$. For this we need some further definitions.

Let V and V' be two Euclidean spaces of dimension n and r respectively. Let $L : V \to V'$ be a linear surjection, then the adjoint operator L^* is injective and its image is $\ker(L)^{\perp}$, so that LL^* is invertible.

Definition 2.4.7. Let $L : V \to V'$ be a surjection, the *pseudo-inverse* (or *Moore–Penrose inverse*) of L is defined as $L^{\dagger} = L^*(LL^*)^{-1}$ from V' to V.

The map $L^{\dagger}$ is the inverse of the restriction of L to $\ker(L)^{\perp}$. It is characterized by the fact that $LL^{\dagger}$ is the identity map of V' and $L^{\dagger}L$ is the orthogonal projection onto $\ker(L)^{\perp}$.

Let $f : M \to \mathbb{R}^r$ be a smooth submersion and assume that $\widetilde{M} = f^{-1}(0)$. Recall that $\nabla^2 f = \nabla^M df$.

Lemma 2.4.8. *Let M be a Riemannian manifold and let $\widetilde{M} \subset M$ be a submanifold of M defined as the zero set of the smooth submersion $f : M \to \mathbb{R}^r$. Let* II *denote the second fundamental form of $\widetilde{M} \subset M$. Then,*

$$\forall x \in \widetilde{M}, \quad \mathrm{II}(x) = (d_xf)^{\dagger} \circ \nabla^2_x f,$$

where $\nabla^2_x f$ is implicitly restricted to $T_x\widetilde{M}$.

Proof. Let $x \in \widetilde{M}$, since $\mathrm{II}(x)$ and $(d_xf)^{\dagger}$ take values in $T_x\widetilde{M}^{\perp} = \ker(d_xf)^{\perp}$, we only need to prove that $d_xf \circ \mathrm{II}(x) = \nabla^2_x f$. Let X and Y be two vector fields on $\widetilde{M}$. The map $df \cdot Y$ vanishes uniformly on $\widetilde{M}$, hence:

$$d_x(df \cdot Y) \cdot X = (\nabla^M_X df)_x \cdot Y + d_xf \cdot (\nabla^M_X Y) = 0.$$

Then, using equation (2.4.5) and $\ker(d_xf) = T_x\widetilde{M}$,

$$(d_xf \circ \mathrm{II}(x))(X,Y) = -d_xf \cdot (\nabla^M_X Y)^{\perp} = -d_xf \cdot (\nabla^M_X Y) = (\nabla^M_X df)_x \cdot Y = \nabla^2_x f(X,Y). \quad \square$$

Proposition 2.4.9. *Let (M, g) be a closed Riemannian manifold of dimension n and R its Riemann curvature. Let $f : M \to \mathbb{R}^r$ be a smooth submersion and $Z_f = f^{-1}(0)$. We denote by $|\mathrm{d}V_f|$ the Riemannian measure on Z_f and by R_f its Riemann curvature.*

If $n - r$ is even, the Euler characteristic of Z_f is:

$$\chi(Z_f) = \frac{1}{(2\pi)^m m!} \int_{x \in Z_f} \operatorname{Tr}\left(R_f(x)^{\wedge m}\right) |\mathrm{d}V_f|\,, \tag{2.4.6}$$

where $m = \frac{n-r}{2}$. Furthermore, for all $x \in Z_f$,

$$R_f(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\left\langle \nabla_x^2 f\,, (d_x f)^{\dagger *}(U)\right\rangle^{\wedge 2}\right], \tag{2.4.7}$$

where U is a standard Gaussian vector in $T_x M$.

Proof. First, we apply the Chern–Gauss–Bonnet theorem 2.4.3 to Z_f, which gives (2.4.6). Then let $x \in Z_f$ and $U \sim \mathcal{N}(0, \mathrm{Id})$ in $T_x M$, by Proposition 2.4.6,

$$R_f(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\langle \mathrm{II}_f(x)\,, U\rangle^{\wedge 2}\right],$$

where II_f is the second fundamental form of $Z_f \subset M$. We conclude by Lemma 2.4.8. $\square$

Proposition 2.4.9 is also true for zero sets of sections. Let s be a section of some rank r vector bundle over M that vanishes transversally and Z_s be its zero set. Let $|\mathrm{d}V_s|$ denote the Riemannian measure on Z_s and R_s denote its Riemann tensor. As above, we can apply Theorem 2.4.3, so that:

$$\chi(Z_s) = \frac{1}{(2\pi)^m m!} \int_{x \in Z_s} \operatorname{Tr}\left(R_s(x)^{\wedge m}\right) |\mathrm{d}V_s|\,. \tag{2.4.8}$$

The result of Proposition 2.4.6 is still valid for Z_s. Besides, the same proof as in Lemma 2.4.8 shows that, for any connection ∇^d, the second fundamental form II_s of Z_s satisfies:

$$\forall x \in Z_s, \quad \mathrm{II}_s(x) = (\nabla_x^d s)^\dagger \circ \nabla_x^{2,d} s.$$

Remark 2.4.10. This is not surprising since the terms of this equality do not depend on a choice of connection and the result in a trivialization is given by Lemma 2.4.8.

Finally, for every connection ∇^d and every $x \in Z_s$, we get:

$$R_s(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\left\langle \nabla_x^{2,d} s\,, (\nabla_x^d s)^{\dagger *}(U)\right\rangle^{\wedge 2}\right], \tag{2.4.9}$$

where U is a standard Gaussian vector in $T_x M$, as in (2.4.7).

2.5 Proofs of the main theorems

We now set to prove the main theorems. The proofs will be detailed in the Riemannian case but only sketched in the real algebraic one, since they are essentially the same.

2.5.1 The Kac–Rice formula

First, we state the celebrated Kac–Rice formula, which is one of the key ingredients in our proofs. This formula is proved in [BSZ01, thm. 4.2], see also [AW09, chap. 6]. For the reader's convenience, we include a proof in Appendix 2.C.

Definition 2.5.1. Let $L : V \to V'$ be a linear map between Euclidean vector spaces. We denote by $\left|\det^\perp(L)\right|$ the *Jacobian* of L:

$$\left|\det{}^\perp(L)\right| = \sqrt{\det(LL^*)},$$

where $L^* : V' \to V$ is the adjoint operator of L.

Remark 2.5.2. If L is not onto then $\left|\det^\perp(L)\right| = 0$. Else, let A be the matrix of the restriction of L to $\ker(L)^\perp$ in any orthonormal basis of $\ker(L)^\perp$ and V', then we have $\left|\det^\perp(L)\right| = |\det(A)|$.

As in section 2.2, we consider a closed Riemannian manifold M of dimension n and a subspace $V \subset \mathcal{C}^\infty(M, \mathbb{R}^r)$ of dimension N (recall that $1 \leqslant r \leqslant n$). We assume that V is 0-ample, in the sense of section 2.2.2, so that

$$\Sigma = \{(f, x) \in V \times M \mid f(x) = 0\}$$

is a submanifold of codimension r of $V \times M$. Let f be a standard Gaussian vector in V. Then Z_f is almost surely a smooth submanifold of codimension r of M (see section 2.2.2). Recall that E denotes both the Schwartz kernel of V and the covariance function of $(f(x))_{x\in M}$.

Theorem 2.5.3 (Kac–Rice formula)**.** *Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function, then*

$$\mathbb{E}\left[\int_{x\in Z_f} \phi(f,x)\,|\mathrm{d}V_f|\right] = \frac{1}{(2\pi)^{\frac{r}{2}}}\int_{x\in M} \frac{\mathbb{E}\left[\phi(f,x)\left|\det^\perp(d_xf)\right| \,\middle|\, f(x)=0\right]}{\sqrt{\det E(x,x)}}\,|\mathrm{d}V_M|\,,$$

whenever one of these integrals is well-defined.

The expectation on the right-hand side is to be understood as the conditional expectation of $\phi(f,x)\left|\det^\perp(d_xf)\right|$ given $f(x) = 0$. By $\det(E(x,x))$, we mean the determinant of the matrix of the bilinear form $E(x,x)$ in any orthonormal basis of $\mathbb{R}^r$.

2.5.2 Proof of Theorem 2.1.1

We start with the expectation of the volume, which is the toy-model for this kind of computations. In this case, the proof is closely related to [BSZ00, BSZ01], in a slightly different setting. The first step is to apply Kac–Rice formula above with $\phi : (f,x) \mapsto 1$. We get:

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \frac{1}{(2\pi)^{\frac{r}{2}}}\int_{x\in M} \frac{1}{\sqrt{\det(E(x,x))}}\mathbb{E}\left[\left|\det^\perp(d_xf)\right| \,\middle|\, f(x)=0\right]|\mathrm{d}V_M|\,. \qquad (2.5.1)$$

Let $x \in M$, then $j^1_x(f) = (f(x), d_xf)$ is a Gaussian vector in $\mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_xM)$ whose distribution only depends on the values of E and its derivatives at (x,x), see Lemma 2.2.4. Thus $\mathbb{E}[\mathrm{Vol}\,(Z_f)]$ will only depend on the values of E and its derivatives along the diagonal as was expected from [BSZ00, thm. 2.2].

The next step is to compute pointwise asymptotics for the integrand on the right-hand side of (2.5.1). We will use Lemma 2.2.4, which describes the distribution of $j^1_x(f)$, and the estimates of section 2.3. Both in the Riemannian and the algebraic settings, the pointwise asymptotic turns out to be universal: it does not depend on x or even on the ambient manifold. This is because the distribution of $j^1_x(f)$ is determined by the asymptotics of section 2.3 which are universal.

We now specify to the case of Riemannian random waves (see section 2.2.5), that is $V = (V_\lambda)^r$ for some non-negative λ. Recall that $(V_\lambda)^r$ is 0-ample (Lemma 2.2.7) so that equation (2.5.1) is valid in this case. Let $x \in M$, by Lemma 2.2.6 and (2.3.1) we have:

$$\det\left(E_\lambda(x,x)\right) = \left(e_\lambda(x,x)\right)^r = \left(\gamma_0 \lambda^{\frac{n}{2}}\right)^r \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right). \tag{2.5.2}$$

Then we want to estimate the conditional expectation in (2.5.1). Before going further, the asymptotics of section 2.3.1 suggest to consider the scaled variables:

$$(t_\lambda, L_\lambda) = \left(\frac{1}{\sqrt{\gamma_0}\lambda^{\frac{n}{4}}} f(x), \frac{1}{\sqrt{\gamma_1}\lambda^{\frac{n+2}{4}}} d_x f\right) \tag{2.5.3}$$

instead of $j^1_x(f)$. This is a centered Gaussian vector whose variance is determined by (2.5.3). Besides, by Definition 2.5.1, the Jacobian is homogeneous of degree r for linear maps taking values in $\mathbb{R}^r$, so that:

$$\mathbb{E}\left[\left|\det{}^\perp(d_x f)\right| \,\middle|\, f(x) = 0\right] = (\gamma_1)^{\frac{r}{2}} \lambda^{\frac{r(n+2)}{4}} \mathbb{E}\left[\left|\det{}^\perp(L_\lambda)\right| \,\middle|\, t_\lambda = 0\right]. \tag{2.5.4}$$

Lemma 2.5.4. *For every $x \in M$, we have:*

$$\mathbb{E}\left[\left|\det{}^\perp(L_\lambda)\right| \,\middle|\, t_\lambda = 0\right] = (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right),$$

where the error term does not depend on the point $x \in M$.

We postpone the proof of this lemma for now and conclude the proof of Theorem 2.1.1. By (2.5.2), (2.5.4) and Lemma 2.5.4, for every $x \in M$,

$$\frac{1}{(2\pi)^{\frac{r}{2}}\sqrt{\det(E(x,x))}} \mathbb{E}\left[\left|\det{}^\perp(d_x f)\right| \,\middle|\, f(x) = 0\right] = \left(\frac{\gamma_1}{\gamma_0}\lambda\right)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right),$$

where the error term does not depend on the point $x \in M$. By (2.3.7) this equals:

$$\left(\frac{\lambda}{n+2}\right)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right).$$

Plugging this into equation (2.5.1) gives Theorem 2.1.1.

Remark 2.5.5. The same proof shows that, for any continuous function $\phi : M \to \mathbb{R}$, we have:

$$\mathbb{E}\left[\int_{Z_f} \phi \left|\mathrm{d}V_f\right|\right] = \left(\frac{\lambda}{n+2}\right)^{\frac{r}{2}} \left(\int_M \phi \left|\mathrm{d}V_M\right|\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} + O(\lambda^{\frac{r-1}{2}}).$$

Hence,

$$\left(\frac{n+2}{\lambda}\right)^{\frac{r}{2}} \mathbb{E}[\left|\mathrm{d}V_f\right|] \xrightarrow[\lambda \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left|\mathrm{d}V_M\right|$$

in the sense of the weak convergence of measures.

We still have to prove Lemma 2.5.4. For this, we need to compute the variance of L_λ given $t_\lambda = 0$. Let $(x_1, \dots, x_n)$ be normal coordinates centered at x, and $(\zeta_1, \dots, \zeta_r)$ denote the canonical basis of $\mathbb{R}^r$. We equip $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M)$ with the orthonormal basis:

$$(\zeta_1, \dots, \zeta_r, \zeta_1 \otimes dx_1, \dots, \zeta_1 \otimes dx_n, \zeta_2 \otimes dx_1, \dots, \zeta_2 \otimes dx_n, \dots, \zeta_r \otimes dx_1, \dots, \zeta_r \otimes dx_n). \tag{2.5.5}$$

Let $\Lambda(\lambda)$ denote the matrix of $\operatorname{Var}(t_\lambda, L_\lambda)$ in this basis. This matrix splits as

$$\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) \end{pmatrix}, \tag{2.5.6}$$

where $\Lambda_{00}(\lambda)$ and $\Lambda_{11}(\lambda)$ are the matrices of $\operatorname{Var}(t_\lambda)$ and $\operatorname{Var}(L_\lambda)$ and $\Lambda_{01}(\lambda) = \Lambda_{10}(\lambda)^{\mathrm{t}}$ is the matrix of $\operatorname{Cov}(t_\lambda, L_\lambda)$. We can decompose further Λ_{10} and Λ_{11} into blocks of size $r \times r$:

$$\Lambda_{10}(\lambda) = \left(\Lambda_{10}^i(\lambda)\right)_{1 \leqslant i \leqslant n}, \qquad \Lambda_{11}(\lambda) = \left(\Lambda_{11}^{i,j}(\lambda)\right)_{1 \leqslant i,j \leqslant n}. \tag{2.5.7}$$

By the Definition (2.5.3) of (t_λ, L_λ), these blocks are obtained by scaling the corresponding blocks in the matrix of $\operatorname{Var}\left(j_x^1(f)\right)$. Lemmas 2.2.4 and 2.2.6 tell us that each one of these blocks is a scalar matrix. Then, using the estimates of section 2.3.1:

$$\Lambda_{00}(\lambda) = \frac{e_\lambda(x,x)}{\gamma_0 \lambda^{\frac{n}{2}}} I_r = I_r + O\left(\lambda^{-\frac{1}{2}}\right), \tag{2.5.8}$$

$$\forall i \in \{1, \dots, n\}, \qquad \Lambda_{10}^i(\lambda) = \frac{\partial_{x_i} e_\lambda(x,x)}{\sqrt{\gamma_0 \gamma_1} \lambda^{\frac{n+1}{2}}} I_r = O\left(\lambda^{-\frac{1}{2}}\right), \tag{2.5.9}$$

$$\text{and } \forall i,j \in \{1, \dots, n\}, \qquad \Lambda_{11}^{i,j}(\lambda) = \frac{\partial_{x_i} \partial_{y_j} e_\lambda(x,x)}{\gamma_1 \lambda^{\frac{n}{2}+1}} I_r = \begin{cases} I_r + O\left(\lambda^{-\frac{1}{2}}\right) & \text{if } i = j, \\ O\left(\lambda^{-\frac{1}{2}}\right) & \text{otherwise,} \end{cases} \tag{2.5.10}$$

where I_r stands for the identity matrix of size r. Thus $\Lambda(\lambda) = I_{r(n+1)} + O\left(\lambda^{-\frac{1}{2}}\right)$ and, by Corollary 2.A.11, the distribution of L_λ conditioned on $t_\lambda = 0$ is a centered Gaussian with variance operator $\widetilde{\Lambda}(\lambda) = \operatorname{Id} + O\left(\lambda^{-\frac{1}{2}}\right)$. Note that these estimates do not depend on x or our choices of coordinates.

Proof of Lemma 2.5.4. Let $\widetilde{L}_\lambda \sim \mathcal{N}(0, \widetilde{\Lambda}(\lambda))$ in $\mathbb{R}^r \otimes T_x^*M$. For λ large enough, $\widetilde{\Lambda}(\lambda)$ is non-singular, and we have:

$$\begin{aligned} \mathbb{E}\left[\left|\det{}^\perp(L_\lambda)\right| \,\middle|\, t_\lambda = 0\right] &= \mathbb{E}\left[\left|\det{}^\perp\left(\widetilde{L}_\lambda\right)\right|\right] \\ &= \frac{1}{(2\pi)^{\frac{nr}{2}} \sqrt{\det(\widetilde{\Lambda}(\lambda))}} \int \left|\det{}^\perp(L)\right| \exp\left(-\frac{1}{2}\left\langle \widetilde{\Lambda}(\lambda)^{-1} L \,, L\right\rangle\right) \mathrm{d}L, \end{aligned} \tag{2.5.11}$$

where $\mathrm{d}L$ stands for the Lebesgue measure on $\mathbb{R}^r \otimes T_x^*M$. Beware that we see L as a linear map in the term $\left|\det^\perp(L)\right|$ but as a vector in $\widetilde{\Lambda}(\lambda)^{-1} L$. The latter is not a composition.

Then $\widetilde{\Lambda}(\lambda) = \operatorname{Id} + O\left(\lambda^{-\frac{1}{2}}\right)$, so that $\left\|\widetilde{\Lambda}(\lambda)^{-1} - \operatorname{Id}\right\|$ is bounded by $\frac{C}{\sqrt{\lambda}}$ for some positive C. Hence, for all $L \in \mathbb{R}^r \otimes T_x^*M$,

$$\left|\left\langle \left(\widetilde{\Lambda}(\lambda)^{-1} - \operatorname{Id}\right) L \,, L\right\rangle\right| \leqslant \frac{C}{\sqrt{\lambda}} \left\|L\right\|^2,$$

and by the mean value theorem,

$$\left|\exp\left(-\frac{1}{2}\left\langle\left(\widetilde{\Lambda}(\lambda)^{-1}-\mathrm{Id}\right)L\,,L\right\rangle\right)-1\right|\leqslant\frac{C}{2\sqrt{\lambda}}\left\|L\right\|^2\exp\left(\frac{C}{2\sqrt{\lambda}}\left\|L\right\|^2\right).$$

Then,

$$\begin{aligned}\left|\int\left|\det{}^{\perp}(L)\right|\left(\exp\left(-\frac{1}{2}\left\langle\widetilde{\Lambda}(\lambda)^{-1}L\,,L\right\rangle\right)-\exp\left(-\frac{\left\|L\right\|^2}{2}\right)\right)\mathrm{d}L\right|&\\ \leqslant\frac{C}{2\sqrt{\lambda}}\int\left|\det{}^{\perp}(L)\right|\left\|L\right\|^2\exp\left(-\frac{\left\|L\right\|^2}{2}\left(1-\frac{C}{\sqrt{\lambda}}\right)\right)\mathrm{d}L.&\quad(2.5.12)\end{aligned}$$

The integral on the right-hand side of (2.5.12) converges to some finite limit as $\lambda\to+\infty$ by Lebesgue's dominated convergence theorem, so that:

$$\int\left|\det{}^{\perp}(L)\right|\exp\left(-\frac{1}{2}\left\langle\widetilde{\Lambda}(\lambda)^{-1}L\,,L\right\rangle\right)\mathrm{d}L=\int\left|\det{}^{\perp}(L)\right|e^{-\frac{1}{2}\left\|L\right\|^2}\,\mathrm{d}L+O\left(\lambda^{-\frac{1}{2}}\right).\tag{2.5.13}$$

Since $\det\left(\widetilde{\Lambda}(\lambda)\right)=1+O\left(\lambda^{-\frac{1}{2}}\right)$, by (2.5.11), (2.5.13) we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}(L_\lambda)\right|\,\middle|\,t_\lambda=0\right]=\mathbb{E}\left[\left|\det{}^{\perp}(L)\right|\right]+O\left(\lambda^{-\frac{1}{2}}\right),$$

where L is a standard Gaussian vector in $\mathbb{R}^r\otimes T_x^*M$. The result of the lemma is given by Lemma 2.A.14. □

2.5.3 Proof of Theorem 2.1.3

We now consider the real algebraic setting described in section 2.2.6. The proof goes along the same lines as above. Recall that $\mathcal{X}$ is a complex projective manifold of dimension n, equipped with a rank r holomorphic vector bundle $\mathcal{E}$ and an ample holomorphic line bundle $\mathcal{L}$, and that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. We are interested in the volume of the real zero set Z_s of a standard Gaussian section s in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$.

By Corollary 2.3.10, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample for d large enough, so that we can apply Kac–Rice formula (Theorem 2.5.3) with $\phi:(s,x)\mapsto 1$, as in the Riemannian case. Note that we have to use the incidence manifold Σ_d defined by (2.2.17) here. As in (2.5.1), we get:

$$\mathbb{E}[\mathrm{Vol}\left(Z_s\right)]=\frac{1}{(2\pi)^{\frac{r}{2}}}\int_{x\in\mathbb{R}\mathcal{X}}\frac{1}{\sqrt{\det(E_d(x,x))}}\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla_x^d s\right)\right|\,\middle|\,s(x)=0\right]\left|\mathrm{d}V_{\mathbb{R}\mathcal{X}}\right|,\tag{2.5.14}$$

where ∇^d is any real connection on $\mathcal{E}\otimes\mathcal{L}^d$.

Let $x\in\mathbb{R}\mathcal{X}$ and $(x_1,\dots,x_n)$ be real holomorphic coordinates around x such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. Let $(\zeta_1^d,\dots,\zeta_r^d)$ be an orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. This yields an orthonormal basis of $\mathcal{J}_x^2(\mathcal{E}\otimes\mathcal{L}^d)$ similar to (2.5.5).

The value of $\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla_x^d s\right)\right|\,\middle|\,s(x)=0\right]$ does not depend on the choice of ∇^d, since $\nabla_x^d s$ does not depend on ∇^d when $s(x)=0$. We choose a connection that satisfies the conditions of section 2.3.3, in order to compute the pointwise asymptotic of this quantity.

The estimates of Proposition 2.3.8 suggest to consider the scaled variables:

$$(t_d,L_d)=\left(\sqrt{\frac{\pi^n}{d^n}}s(x),\sqrt{\frac{\pi^n}{d^{n+1}}}\nabla_x^d s\right).\tag{2.5.15}$$

Then, (t_d, L_d) is a centered Gaussian vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$, and the matrix of $\mathrm{Var}(t_d, L_d)$ in the basis described above is $I_{r(n+1)} + O(d^{-1})$. This is proved by the same kind of computation as in the Riemannian case, using the estimates of Proposition 2.3.8. The distribution of L_d given $t_d = 0$ is then a centered Gaussian with variance operator:

$$\widetilde{\Lambda}(d) = \mathrm{Id} + O(d^{-1}).$$

As in the previous section (cf. Lemma 2.5.4), for every $x \in \mathbb{R}\mathcal{X}$,

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla^d_x s\right)\right| \,\middle|\, s(x) = 0\right] &= \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}} \mathbb{E}\left[\left|\det{}^{\perp}(L_d)\right| \,\middle|\, t_d = 0\right] \\ &= \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}} (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} \left(1 + O\big(d^{-1}\big)\right).\end{aligned} \tag{2.5.16}$$

Besides, the estimate (2.3.13) shows that:

$$\det\left(E_d(x,x)\right) = \left(\frac{d}{\pi}\right)^{rn} \left(1 + O\big(d^{-1}\big)\right), \tag{2.5.17}$$

Finally, by (2.5.14), (2.5.16) and (2.5.17), we have proved Theorem 2.1.3.

As for Riemannian random waves, the same proof shows that for any continuous function $\phi : \mathbb{R}\mathcal{X} \to \mathbb{R}$, we have:

$$\mathbb{E}\left[\int_{Z_s} \phi \left|\mathrm{d}V_s\right|\right] = d^{\frac{r}{2}} \left(\int_{\mathbb{R}\mathcal{X}} \phi \left|\mathrm{d}V_{\mathbb{R}\mathcal{X}}\right|\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} + O\left(d^{\frac{r}{2}-1}\right).$$

Hence,

$$\left(d^{-\frac{r}{2}}\right) \mathbb{E}[|\mathrm{d}V_s|] \xrightarrow[\lambda \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} \left|\mathrm{d}V_M\right|$$

in the sense of the weak convergence of measures.

2.5.4 Proof of Theorem 2.1.2

In this section we compute the expected Euler characteristic of our random submanifolds. The proof is basically the same as in the volume case: apply Kac–Rice formula then compute a pointwise asymptotic for the conditional expectation that appears in Theorem 2.5.3. Only, this time, we apply Kac–Rice formula to a quantity $\phi(f,x)$ that really depends on the couple $(f,x) \in \Sigma$. This makes the computations a bit more complex. Luckily, $\phi(f,x)$ only depends on the 2-jet of f at x, so we can still make pointwise computations.

Consider first the general setting of sections 2.2.1 to 2.2.4: f is a standard Gaussian vector in the finite-dimensional subspace $V \subset \mathcal{C}^\infty(M, \mathbb{R}^r)$. We assume that V is 0-ample, so that for almost every $f \in V$, Z_f is a closed submanifold of dimension $n - r$.

If $n - r$ is odd, then $\chi(Z_f) = 0$ almost surely (see Remark 2.4.4). From now on, we assume that $n - r$ is even and set $m = \frac{n-r}{2}$. If $n = r$, then Z_f is almost surely a finite set, and $\chi(Z_f) = \mathrm{Vol}\left(Z_f\right)$ is just the cardinal of Z_f. In this case Theorems 2.1.2 and 2.1.1 coincide, so we need only consider the case $r < n$ in the sequel.

We denote by R the Riemann curvature of the ambient manifold M and, for any $f \in V$, we denote by R_f the Riemann curvature of Z_f. By Proposition 2.4.9 and Kac–Rice formula (Theorem 2.5.3),

$$\begin{aligned}\mathbb{E}[\chi(Z_f)] &= \mathbb{E}\left[\frac{1}{(2\pi)^m m!} \int_{Z_f} \mathrm{Tr}\left((R_f)^{\wedge m}\right) \left|\mathrm{d}V_f\right|\right] \\ &= \frac{1}{m!(2\pi)^{\frac{n}{2}}} \int_{x \in M} \frac{1}{\sqrt{\det(E(x,x))}} \mathbb{E}\left[\left|\det{}^{\perp}(d_x f)\right| \mathrm{Tr}(R_f(x)^{\wedge m}) \,\middle|\, f(x) = 0\right] \left|\mathrm{d}V_M\right|.\end{aligned} \tag{2.5.18}$$

Moreover, for all $x \in M$, let $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM be independent of f. Then, Proposition 2.4.9 gives:

$$R_f(x) = R(x) + \frac{1}{2}\mathbb{E}_U\left[\left\langle \nabla_x^2 f \,, d_x f^{\dagger *}(U)\right\rangle^{\wedge 2}\right], \tag{2.5.19}$$

where the notation $\mathbb{E}_U[\,\cdot\,]$ means that we only take the expectation with respect to the variable U. Here and in everything that follows, $R(x)$ and $\nabla_x^2 f$ are implicitly restricted to $\ker(d_x f) = T_x Z_f$.

As in the volume case, the next step is to compute the pointwise asymptotic for the integrand in the last term of (2.5.18). By (2.5.19), it only depends on $R(x)$ and the distribution of $j_x^2(f)$ which is characterized by Lemma 2.2.4. This shows that the expected Euler characteristic only depends on R and the values of E and its derivatives (up to order 2 in each variable) along the diagonal in $M \times M$. It turns out that, both in the Riemannian and the algebraic cases, the pointwise asymptotic is universal and no longer depends on R.

Focusing on random waves, $V = (V_\lambda)^r$ and we already know from Lemma 2.2.7 that $(V_\lambda)^r$ is 0-ample. Let $x \in M$, recall that $\det(E_\lambda(x,x)) = \left(\gamma_0 \lambda^{\frac{n}{2}}\right)^r \left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right)$ (see (2.5.2)). Then, the main task is to estimate the conditional expectation in (2.5.18).

We consider the scaled variables:

$$(t_\lambda, L_\lambda, S_\lambda) = \left(\frac{1}{\sqrt{\gamma_0}\lambda^{\frac{n}{4}}} f(x), \frac{1}{\sqrt{\gamma_1}\lambda^{\frac{n+2}{4}}} d_x f, \frac{1}{\sqrt{\gamma_2}\lambda^{\frac{n+4}{4}}} \nabla_x^2 f\right) \tag{2.5.20}$$

in $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M \oplus \mathrm{Sym}(T_x^*M))$. By Lemma 2.2.4, $(t_\lambda, L_\lambda, S_\lambda)$ is a centered Gaussian vector. We denote by $(\widetilde{L}_\lambda, \widetilde{S}_\lambda)$ a random variable in $\mathbb{R}^r \otimes (T_x^*M \oplus \mathrm{Sym}(T_x^*M))$ distributed as (L_λ, S_λ) given $t_\lambda = 0$. Again, $(\widetilde{L}_\lambda, \widetilde{S}_\lambda)$ is a centered Gaussian, see Corollary 2.A.11.

Let $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM be independent of f (hence of $j_x^2(f)$) and $(\widetilde{L}_\lambda, \widetilde{S}_\lambda)$. Then, by (2.5.19), (2.5.20) and (2.3.7):

$$\begin{aligned}
R_f(x) &= R(x) + \frac{1}{2}\mathbb{E}_U\left[\left\langle \nabla_x^2 f \,, d_x f^{\dagger *}(U)\right\rangle^{\wedge 2}\right] \\
&= R(x) + \frac{1}{2}\mathbb{E}_U\left[\left\langle \sqrt{\gamma_2}\lambda^{\frac{n+4}{4}} S_\lambda \,, \frac{L_\lambda^{\dagger *}(U)}{\sqrt{\gamma_1}\lambda^{\frac{n+2}{4}}}\right\rangle^{\wedge 2}\right] \\
&= R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle S_\lambda \,, L_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right].
\end{aligned}$$

Besides, $\left|\det^\perp(d_x f)\right| = (\gamma_1)^{\frac{r}{2}} \lambda^{\frac{r(n+2)}{4}} \left|\det^\perp(L_\lambda)\right|$, so that,

$$\begin{aligned}
&\mathbb{E}\left[\left|\det^\perp(d_x f)\right| \mathrm{Tr}(R_f(x)^{\wedge m}) \,\middle|\, f(x) = 0\right] \\
&\quad = (\gamma_1)^{\frac{r}{2}} \lambda^{\frac{r(n+2)}{4}} \times \\
&\qquad\qquad \mathbb{E}\left[\left|\det^\perp(L_\lambda)\right| \mathrm{Tr}\left(\left(R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle S_\lambda \,, L_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right) \,\middle|\, t_\lambda = 0\right] \\
&\quad = (\gamma_1)^{\frac{r}{2}} \lambda^{\frac{r(n+2)}{4}} \mathbb{E}\left[\left|\det^\perp\left(\widetilde{L}_\lambda\right)\right| \mathrm{Tr}\left(\left(R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right].
\end{aligned} \tag{2.5.21}$$

To conclude the proof, we will use the following lemmas.

Lemma 2.5.6. *The random vectors* $\left(\widetilde{L}_\lambda, \widetilde{S}_\lambda\right)$ *converge in distribution to a Gaussian vector* (L,S) *as* $\lambda \to +\infty$. *Let* $U \sim \mathcal{N}(0,\mathrm{Id})$ *in* T_xM *be independent of* $\left(\widetilde{L}_\lambda, \widetilde{S}_\lambda\right)$ *and* (L,S), *then:*

$$\mathbb{E}\left[\left|\det^\perp\left(\widetilde{L}_\lambda\right)\right| \operatorname{Tr}\left(\left(R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right]$$
$$= \left(\frac{\lambda}{n+4}\right)^m \frac{m!}{(2m)!}\mathbb{E}\left[\left|\det^\perp(L)\right| \operatorname{Tr}\left(\left\langle S \,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right]\left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right),$$

where the error term is uniform in $x \in M$.

Lemma 2.5.7. *Let* (L,S) *be distributed as the limit of* $\left(\widetilde{L}_\lambda, \widetilde{S}_\lambda\right)$ *and* $U \sim \mathcal{N}(0,\mathrm{Id})$ *in* T_xM *be independent of* (L,S). *We have:*

$$\mathbb{E}\left[\left|\det^\perp(L)\right| \operatorname{Tr}\left(\left\langle S \,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] = \left(-\frac{n+4}{n+2}\right)^m (2\pi)^{\frac{n}{2}}(2m)!\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^{n}\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}.$$

Assuming these lemmas and recalling (2.5.2) and (2.5.21), the integrand in the last term of (2.5.18) equals:

$$(-1)^m m!(2\pi)^{\frac{n}{2}}\left(\frac{\gamma_1}{\gamma_0}\lambda\right)^{\frac{r}{2}}\left(\frac{\lambda}{n+2}\right)^m \frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^{n}\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}\left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right),$$

where the error term is uniform in $x \in M$. Since $r + 2m = n$ and $\frac{\gamma_0}{\gamma_1} = n+2$, we finally get:

$$\mathbb{E}[\chi(Z_f)] = (-1)^m\left(\frac{\lambda}{n+2}\right)^{\frac{n}{2}} \frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^{n}\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}\int_{x\in M}\left(1 + O\left(\lambda^{-\frac{1}{2}}\right)\right)\left|\mathrm{d}V_M\right|,$$

and this is Theorem 2.1.2.

We now have to prove Lemmas 2.5.6 and 2.5.7. For this we will need the following technical result which is a reformulation of [Bü06, prop. 3.12]. The proof of Proposition 2.5.8 is mostly tedious computations and we postpone it until Appendix 2.B.

Proposition 2.5.8. *Let* V *and* V' *be two Euclidean spaces of dimension* n *and* r *respectively, with* $1 \leqslant r \leqslant n$. *Let* $L \in V' \otimes V^*$ *and* $U \in V$ *be independent standard Gaussian vectors. Then,* $L^\dagger$ *is well-defined almost surely and* $\left(\left|\det^\perp(L)\right|, (L^\dagger)^*U\right)$ *has the same distribution as*

$$\left(\|X_n\|\,\|X_{n-1}\|\cdots\|X_{n-r+1}\|\,, \frac{U'}{\|X_{n-r+1}\|}\right),$$

where $U' \in V'$, $X_p \in \mathbb{R}^p$ *for all* $p \in \{n-r+1,\dots,n\}$ *and* $U', X_n, \dots, X_{n-r+1}$ *are globally independent standard Gaussian vectors.*

We start by computing the variance of $(t_\lambda, L_\lambda, S_\lambda)$. As in section 2.5.2, we choose normal coordinates $(x_1,\dots,x_n)$ centered at x and denote by $(\zeta_1,\dots,\zeta_r)$ the canonical basis of $\mathbb{R}^r$. For any i and j such that $1 \leqslant i < j \leqslant n$, we set $dx_{ij} = (dx_i \otimes dx_j + dx_j \otimes dx_i)$ and $dx_{ii} = dx_i \otimes dx_i$. We complete the basis of $\mathcal{J}_x^1(\mathbb{R}^r)$ given in (2.5.5) into an orthonormal basis of $\mathcal{J}_x^2(\mathbb{R}^r)$ by adding the following elements (in this order) at the end of the list:

$$\begin{aligned}&\zeta_1 \otimes dx_{11}, \dots, \zeta_r \otimes dx_{11}, \zeta_1 \otimes dx_{22}, \dots, \zeta_r \otimes dx_{22}, \dots, \zeta_1 \otimes dx_{nn}, \dots, \zeta_r \otimes dx_{nn},\\ &\quad \zeta_1 \otimes dx_{12}, \dots, \zeta_r \otimes dx_{12}, \zeta_1 \otimes dx_{13}, \dots, \zeta_r \otimes dx_{13}, \dots, \zeta_1 \otimes dx_{1n}, \dots, \zeta_r \otimes dx_{1n},\\ &\qquad \dots, \zeta_1 \otimes dx_{(n-1)n}, \dots, \zeta_r \otimes dx_{(n-1)n}.\end{aligned} \quad (2.5.22)$$

The matrix of $\mathrm{Var}(t_\lambda, L_\lambda, S_\lambda)$ with respect to this basis is:

$$\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) & \Lambda_{02}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) & \Lambda_{12}(\lambda) \\ \Lambda_{20}(\lambda) & \Lambda_{21}(\lambda) & \Lambda_{22}(\lambda) \end{pmatrix},$$

where $\Lambda_{00}(\lambda)$, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ are as in (2.5.6) and, similarly, $\Lambda_{22}(\lambda)$ is the matrix of $\mathrm{Var}(S_\lambda)$, $\Lambda_{02}(\lambda) = \Lambda_{20}(\lambda)^{\mathrm{t}}$ is the matrix of $\mathrm{Cov}(t_\lambda, S_\lambda)$ and $\Lambda_{12}(\lambda) = \Lambda_{21}(\lambda)^{\mathrm{t}}$ is the matrix of $\mathrm{Cov}(L_\lambda, S_\lambda)$. As in (2.5.7), we can decompose further each of these matrices in blocks of size $r \times r$. That is, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ satisfy (2.5.7) and,

$$\begin{aligned} \Lambda_{20}(\lambda) = \left(\Lambda_{20}^{ik}(\lambda)\right)_{1 \leqslant i \leqslant k \leqslant n}, \quad & \Lambda_{21}(\lambda) = \left(\Lambda_{21}^{ik,j}(\lambda)\right)_{\substack{1 \leqslant i \leqslant k \leqslant n \\ 1 \leqslant j \leqslant n}} \\ \text{and} \qquad\qquad & \Lambda_{22}(\lambda) = \left(\Lambda_{22}^{ik,jl}(\lambda)\right)_{\substack{1 \leqslant i \leqslant k \leqslant n \\ 1 \leqslant j \leqslant l \leqslant n}}. \end{aligned} \tag{2.5.23}$$

By definition of $(t_\lambda, L_\lambda, S_\lambda)$, these blocks are obtained by scaling the corresponding blocks in the matrix of $\mathrm{Var}\big(j_x^2(f)\big)$. By Lemmas 2.2.4 and 2.2.6 the matrices in (2.5.23) are scalar matrices. Recalling (2.3.7), we set: $\gamma = -\sqrt{\frac{\gamma_1^2}{\gamma_0\gamma_2}} = -\sqrt{\frac{n+4}{n+2}}$. Then, by (2.5.20) and Theorem 2.3.1 we have:

$$\Lambda_{20}^{ik}(\lambda) = \frac{\partial_{x_i,x_k} e_\lambda(x,x)}{\sqrt{\gamma_0\gamma_2}\lambda^{\frac{n+2}{2}}} I_r = \begin{cases} \gamma I_r + O\left(\lambda^{-\frac{1}{2}}\right) & \text{if } i = k, \\ O\left(\lambda^{-\frac{1}{2}}\right) & \text{if } i \neq k, \end{cases} \tag{2.5.24}$$

$$\Lambda_{21}^{ik,j}(\lambda) = \frac{\partial_{x_i,x_k}\partial_{y_j} e_\lambda(x,x)}{\sqrt{\gamma_1\gamma_2}\lambda^{\frac{n+3}{2}}} I_r = O\left(\lambda^{-\frac{1}{2}}\right), \tag{2.5.25}$$

$$\Lambda_{22}^{ik,jl}(\lambda) = \frac{\partial_{x_i,x_k}\partial_{y_j,y_l} e_\lambda(x,x)}{\gamma_2\lambda^{\frac{n+4}{2}}} I_r = \begin{cases} 3I_r + O\left(\lambda^{-\frac{1}{2}}\right) & \text{if } i = j = k = l, \\ I_r + O\left(\lambda^{-\frac{1}{2}}\right) & \text{if } i = j \neq k = l \\ & \text{or } i = k \neq j = l, \\ O\left(\lambda^{-\frac{1}{2}}\right) & \text{otherwise,} \end{cases} \tag{2.5.26}$$

where I_r denotes the identity matrix of size r. Similar estimates for $\Lambda_{00}(\lambda)$, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ are given by (2.5.8), (2.5.9) and (2.5.10) respectively. Then $\Lambda(\lambda)$ writes by blocks:

$$\Lambda(\lambda) = \left(\begin{array}{c|c|cccc|c} I_r & & \gamma I_r & \gamma I_r & \cdots & \gamma I_r & \\ \hline & I_{nr} & & & & & \\ \hline \gamma I_r & & 3I_r & I_r & \cdots & I_r & \\ \gamma I_r & & I_r & 3I_r & \ddots & \vdots & \\ \vdots & & \vdots & \ddots & \ddots & I_r & \\ \gamma I_r & & I_r & \cdots & I_r & 3I_r & \\ \hline & & & & & & I_{\frac{rn(n-1)}{2}} \end{array}\right) + O\left(\lambda^{-\frac{1}{2}}\right),$$

where the empty blocks are zeros. The distribution of $(\widetilde{L}_\lambda, \widetilde{S}_\lambda)$, that is the distribution of

(L_λ, S_λ) given $t_\lambda = 0$, is a centered Gaussian whose variance matrix is:

$$\widetilde{\Lambda}(\lambda) = \left(\begin{array}{c|cccc|c} I_{nr} & & & & & \\ \hline & \beta_0 I_r & \beta I_r & \cdots & \beta I_r & \\ & \beta I_r & \beta_0 I_r & \ddots & \vdots & \\ & \vdots & \ddots & \ddots & \beta I_r & \\ & \beta I_r & \cdots & \beta I_r & \beta_0 I_r & \\ \hline & & & & & I_{\frac{rn(n-1)}{2}} \end{array}\right) + O\left(\lambda^{-\frac{1}{2}}\right), \tag{2.5.27}$$

where $\beta = 1 - \gamma^2 = -\dfrac{2}{n+2}$ and $\beta_0 = 3 - \gamma^2 = \dfrac{2n+2}{n+2}$ (see Corollary 2.A.11).

Let Λ denote the leading term in (2.5.27). Equation (2.5.27) shows that the random vectors $\left(\widetilde{L}_\lambda, \widetilde{S}_\lambda\right)$ converge in distribution to a random vector $(L, S) \sim \mathcal{N}(0, \Lambda)$ (see Lemma 2.A.12). We have:

$$\det(\Lambda) = (\beta_0 + (n-1)\beta)^r (\beta_0 - \beta)^{r(n-1)} = \left(\frac{4(2n+4)^{n-1}}{(n+2)^n}\right)^r,$$

so that Λ is non-singular, and $\widetilde{\Lambda}(\lambda)$ is non-singular for λ large enough.

Proof of Lemma 2.5.6. We have already seen that $\left(\widetilde{L}_\lambda, \widetilde{S}_\lambda\right)$ converges in distribution, as λ goes to infinity, to $(L, S) \sim \mathcal{N}(0, \Lambda)$. We still have to prove the estimate in the lemma. We have:

$$\begin{aligned}\left(R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m} \\ = \sum_{q=0}^{m} \binom{m}{q} \left(\frac{\lambda}{2(n+4)}\right)^q R(x)^{\wedge(m-q)} \wedge \mathbb{E}_U\left[\left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]^{\wedge q}.\end{aligned}$$

We can apply Lemma 2.4.2 to each term in this sum. This yields:

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}\left(\widetilde{L}_\lambda\right)\right| \operatorname{Tr}\left(\left(R(x) + \frac{\lambda}{2(n+4)}\mathbb{E}_U\left[\left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right] \\ &= \sum_{q=0}^{m} \binom{m}{q} \frac{q!}{(2q)!} \left(\frac{\lambda}{2(n+4)}\right)^q \mathbb{E}\left[\left|\det{}^{\perp}\left(\widetilde{L}_\lambda\right)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right].\end{aligned} \tag{2.5.28}$$

Then it is sufficient to show that, for all $q \in \{0, \dots, m\}$,

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}\left(\widetilde{L}_\lambda\right)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle \widetilde{S}_\lambda \,, \widetilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] \\ &\qquad = \mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S \,, L^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] + O\left(\lambda^{-\frac{1}{2}}\right),\end{aligned} \tag{2.5.29}$$

and that these terms are finite. Then (2.5.28) and (2.5.29) yield the estimate in the lemma.

Let $q \in \{0, \dots, m\}$, then we first show that the principal part on the right-hand side of (2.5.29) is finite. Let $\zeta \in \mathbb{R}^r$, then

$$\mathbb{E}_S\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \langle S \,, \zeta\rangle^{\wedge 2q}\right)\right] \tag{2.5.30}$$

is finite since it is the expectation of some polynomial in the coefficients of S. Thus, (2.5.30) only depends on ζ, and it is an homogeneous polynomial in ζ of degree $2q$.

We assumed U to be independent of (L, S), and the expression (2.5.27) of Λ shows that L and S are independent. Let $U' \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^r$ and $X_{n-r+1}, \dots, X_n$ be standard Gaussian vectors, with $X_p \in \mathbb{R}^p$, such that $U', S, X_{n-r+1}, \dots, X_n$ are globally independent. Applying Proposition 2.5.8, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] \\
&\qquad = \mathbb{E}\left[\frac{\|X_n\| \cdots \|X_{n-r+2}\|}{\|X_{n-r+1}\|^{2q-1}} \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S\,, U'\right\rangle^{\wedge 2q}\right)\right] \\
&\qquad = \mathbb{E}\left[\frac{1}{\|X_{n-r+1}\|^{2q-1}}\right] \mathbb{E}\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S\,, U'\right\rangle^{\wedge 2q}\right)\right] \prod_{p=n-r+2}^{n} \mathbb{E}[\|X_p\|]\,.
\end{aligned} \tag{2.5.31}$$

Since $2q - 1 \leqslant 2m - 1 \leqslant n - r - 1$ and X_{n-r+1} is a standard Gaussian in $\mathbb{R}^{n-r+1}$, $\mathbb{E}\left[\frac{1}{\|X_{n-r+1}\|^{2q-1}}\right] < +\infty$ (see Lemma 2.A.13). The other factors on the right-hand side of (2.5.31) are expectations of polynomials is some standard Gaussian variables, so they are finite.

Then, $\widetilde{\Lambda}(\lambda) = \Lambda + O\left(\lambda^{-\frac{1}{2}}\right)$, and the same kind of computations as in the proof of Lemma 2.5.4 gives (2.5.29), which concludes the proof of Lemma 2.5.6. Note that, since M is compact, $R(x)$ is bounded, independently of x. We need this fact to ensure that the error term in (2.5.29) is independent of x. $\square$

Proof of Lemma 2.5.7. Let $(L, S) \sim \mathcal{N}(0, \Lambda)$ in $\mathbb{R}^r \otimes (T_x^*M \oplus \mathrm{Sym}(T_x^*M))$ and $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM be independent of (L, S). By (2.5.31):

$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(\left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] = \mathbb{E}\left[\frac{\|X_{n-r+2}\| \dots \|X_n\|}{\|X_{n-r+1}\|^{2m-1}}\right] \mathbb{E}\left[\operatorname{Tr}\left(\left\langle S\,, U'\right\rangle^{\wedge 2m}\right)\right], \tag{2.5.32}$$

where $U' \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^r$, $X_p \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^p$ for all p, and $U', S, X_{n-r+1}, \dots, X_n$ are globally independent.

Recall that we are only interested in the restriction of S to $\ker(L) \subset T_xM$. But L and S are independent, as one can see on the expression of Λ (2.5.27), and the distribution of S is invariant under orthogonal transformations of T_xM. Thus, we can consider S restricted to any $2m$-dimensional subspace of T_xM in our computations. For simplicity, we restrict S to V, the span of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}\right)$.

We now compute the term $\mathbb{E}\left[\operatorname{Tr}\left(\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2m}\right)\right]$. By Lemma 2.4.2,

$$\mathbb{E}_S\left[\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2m}\right] = \frac{(2m)!}{2^m m!} \mathbb{E}_S\left[\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2}\right]^{\wedge m}.$$

Assuming that $S_{/V} = \displaystyle\sum_{1 \leqslant i,k \leqslant 2m} S_{ik}\, dx_i \otimes dx_k$, with $S_{ik} \in \mathbb{R}^r$, we have:

$$\left\langle S_{/V}\,, U'\right\rangle = \sum_{1 \leqslant i,k \leqslant 2m} \left\langle S_{ik}\,, U'\right\rangle dx_i \otimes dx_k.$$

By Lemma 2.A.8,

$$\mathbb{E}_S\Big[\big\langle S_{/V}\,,U'\big\rangle^{\wedge 2}\Big] = \sum_{1\leqslant i,j,k,l\leqslant 2m} \mathbb{E}_S\big[\langle S_{ik}\,,U'\rangle\,\langle S_{jl}\,,U'\rangle\big]\,(dx_i\wedge dx_j)\otimes(dx_k\wedge dx_l)$$
$$= \sum_{1\leqslant i,j,k,l\leqslant 2m} \Big\langle U'\,,\Big(\Lambda^{ik,jl}\Big)\,U'\Big\rangle\,(dx_i\wedge dx_j)\otimes(dx_k\wedge dx_l),$$

where we denoted by $\Lambda^{ik,jl}$ the covariance operator of S_{ik} and S_{jl}. Then,

$$\mathbb{E}_S\Big[\big\langle S\,,U'\big\rangle^{\wedge 2m}\Big] = \frac{(2m)!}{2^m m!}\sum_{\substack{1\leqslant i_1,\dots,i_m\leqslant 2m\\ 1\leqslant j_1,\dots,j_m\leqslant 2m\\ 1\leqslant k_1,\dots,k_m\leqslant 2m\\ 1\leqslant l_1,\dots,l_m\leqslant 2m}} \left(\prod_{p=1}^m \Big\langle U'\,,\Big(\Lambda^{i_pk_p,j_pl_p}\Big)\,U'\Big\rangle\right)\times$$
$$(dx_{i_1}\wedge dx_{j_1}\wedge\cdots\wedge dx_{i_m}\wedge dx_{j_m})\otimes(dx_{k_1}\wedge dx_{l_1}\wedge\cdots\wedge dx_{k_m}\wedge dx_{l_m})$$
$$= \frac{(2m)!}{2^m m!}\sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma)\varepsilon(\sigma')\prod_{p=1}^m \Big\langle U'\,,\Big(\Lambda^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)}\Big)\,U'\Big\rangle\,(dx\otimes dx),$$

where $\mathfrak{S}_{2m}$ is the set of permutations of $\{1,\dots,2m\}$, $\varepsilon:\mathfrak{S}_{2m}\to\{-1,1\}$ denotes the signature morphism and $dx = dx_1\wedge\cdots\wedge dx_{2m}$. We get the last line by setting $\sigma(2p-1)=i_p$, $\sigma(2p)=j_p$, $\sigma'(2p-1)=k_p$ and $\sigma'(2p)=l_p$ and reordering the wedge products.

Since our local coordinates are such that $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{2m}}\right)$ is orthonormal at x, we have $\operatorname{Tr}(dx\otimes dx)=\|dx\|^2=1$. Thus,

$$\operatorname{Tr}\left(\mathbb{E}_S\Big[\big\langle S\,,U'\big\rangle^{\wedge 2m}\Big]\right) = \frac{(2m)!}{2^m m!}\sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\prod_{p=1}^m \Big\langle U'\,,\Big(\Lambda^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)}\Big)\,U'\Big\rangle. \tag{2.5.33}$$

By equation (2.5.27), for any σ, $\sigma'\in\mathfrak{S}_{2m}$ and for any $p\in\{1,\dots,m\}$,

$$\Lambda^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)} = K(p,\sigma,\sigma')\,\mathrm{Id},$$

where

$$K(p,\sigma,\sigma') = \begin{cases} \beta & \text{if } \sigma(2p-1)=\sigma'(2p-1) \text{ and } \sigma(2p)=\sigma'(2p),\\ 1 & \text{if } \sigma(2p-1)=\sigma'(2p) \text{ and } \sigma(2p)=\sigma'(2p-1),\\ 0 & \text{otherwise.}\end{cases} \tag{2.5.34}$$

Note that $K(p,\sigma,\sigma')=K(p,\mathrm{id},\sigma^{-1}\circ\sigma')$, where id stands for the identity permutation. Then, setting $\tau=\sigma^{-1}\circ\sigma'$,

$$\sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\prod_{p=1}^m \Big\langle U'\,,\Big(\widetilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)}\Big)\,U'\Big\rangle$$
$$\begin{aligned} &= \sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\,\big\|U'\big\|^{2m}\prod_{p=1}^m K(p,\sigma,\sigma')\\ &= (2m)!\,\big\|U'\big\|^{2m}\sum_{\tau\in\mathfrak{S}_{2m}}\varepsilon(\tau)\prod_{p=1}^m K(p,\mathrm{id},\tau).\end{aligned} \tag{2.5.35}$$

From the Definition (2.5.34) of $K(p, \mathrm{id}, \tau)$, we get that $\prod_{p=1}^{m} K(p, \mathrm{id}, \tau) \neq 0$ if and only if τ is a product of transpositions of the type $((2p-1)\ (2p))$. Now, if $I \subset \{1, \dots, m\}$ and $\tau = \prod_{p \in I}((2p-1)\ (2p))$, we have:

$$\prod_{p=1}^{m} K(p, \mathrm{id}, \tau) = \beta^{m-|I|} \qquad \text{and} \qquad \varepsilon(\tau) = (-1)^{|I|},$$

where $|I|$ stands for the cardinal of I. Thus,

$$\begin{aligned}\sum_{\tau \in \mathfrak{S}_{2m}} \varepsilon(\tau) \prod_{p=1}^{m} K(p, \mathrm{id}, \tau) = \sum_{I \subset \{1,\dots,m\}} (-1)^{|I|} \beta^{m-|I|} &= \sum_{p=1}^{m} \binom{m}{p} (-1)^p \beta^{m-p} \\ &= (\beta - 1)^m = (-1)^m \left(\frac{n+4}{n+2}\right)^m .\end{aligned} \tag{2.5.36}$$

Finally, by equations (2.5.33), (2.5.35) and (2.5.36),

$$\mathbb{E}\left[\mathrm{Tr}\left(\left\langle S\,, U'\right\rangle^{\wedge 2m}\right)\right] = (-1)^m \left(\frac{n+4}{n+2}\right)^m \frac{((2m)!)^2}{2^m m!} \mathbb{E}\left[\left\|U'\right\|^{2m}\right]. \tag{2.5.37}$$

Then, by (2.5.32), (2.5.37) and Lemma 2.A.13,

$$\begin{aligned}&\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \mathrm{Tr}\left(\left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] \\ &\qquad = (-1)^m \left(\frac{n+4}{n+2}\right)^m \frac{((2m)!)^2}{2^m m!} (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{1}\right)} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} \frac{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}.\end{aligned}$$

We conclude the proof of Lemma 2.5.7 by computing:

$$(2\pi)^{\frac{r}{2}} \frac{(2m)!}{2^m m!} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{2} = (2\pi)^{\frac{r}{2}} \frac{2^m \Gamma\left(m + \frac{1}{2}\right)}{\sqrt{\pi}} \frac{\pi^{m+\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)} = (2\pi)^{\frac{n}{2}}. \qquad \square$$

2.5.5 Proof of Theorem 2.1.4

Let us now adapt the proof of the previous section to the case of real algebraic submanifolds. Once again, we need only consider the case where $r < n$ and $n - r$ is even. The framework is the same as in sections 2.2.6 and 2.5.3.

We already know that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample by Corollary 2.3.10. We also have the estimate (2.5.17) for $\det(E_d)$ along the diagonal, where E_d is the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. As in the Riemannian case, we use (2.4.8) and Kac–Rice formula:

$$\mathbb{E}[\chi(Z_s)] = \frac{1}{m!(2\pi)^{\frac{n}{2}}} \int_{x \in \mathbb{R}\mathcal{X}} \frac{\mathbb{E}\left[\left|\det{}^{\perp}(\nabla^d_x s)\right| \mathrm{Tr}\left(R_s(x)^{\wedge m}\right) \,\middle|\, s(x) = 0\right]}{\sqrt{\det(E_d(x,x))}} \left|\mathrm{d}V_{\mathbb{R}\mathcal{X}}\right|, \tag{2.5.38}$$

where ∇^d is any real connection on $(\mathcal{E} \otimes \mathcal{L}^d)$ and R_s denotes the Riemann tensor of Z_s.

We need to compute the conditional expectation in (2.5.38) for some fixed $x \in \mathbb{R}\mathcal{X}$. Since it does not depend on our choice of connection, we will use one that is adapted to x as in sections 2.3.3 and 2.5.3. Let $x \in \mathbb{R}\mathcal{X}$, then we consider the scaled variables:

$$(t_d, L_d, S_d) = \left(\sqrt{\frac{\pi^n}{d^n}} s(x), \sqrt{\frac{\pi^n}{d^{n+1}}} \nabla^d_x s, \sqrt{\frac{\pi^n}{d^{n+2}}} \nabla^{2,d}_x s\right), \tag{2.5.39}$$

and $(\widetilde{L}_d,\widetilde{S}_d) \in \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes (T_x^*\mathbb{R}\mathcal{X} \oplus \operatorname{Sym}(T_x^*\mathbb{R}\mathcal{X}))$ distributed as (L_d,S_d) given $t_d=0$.

As in section 2.5.3, let $(x_1,\dots,x_n)$ be real holomorphic coordinates centered at x and such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. Let $(\zeta_1^d,\dots,\zeta_r^d)$ be an orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. We get an orthonormal basis of $\mathcal{J}_x^2(\mathcal{E}\otimes\mathcal{L}^d)$ similar to the one defined by (2.5.5) and (2.5.22).

We compute the matrix $\Lambda(d)$ of $\operatorname{Var}(t_d,L_d,S_d)$ in this basis. For this we use the estimates for the blocks of $\Lambda(d)$ given by Proposition 2.3.8 and (2.5.39). Namely, for all i, j, k and $l\in\{1,\dots,n\}$, with $i\leqslant k$ and $j\leqslant l$,

$$\Lambda_{20}^{ik}(d) = O\big(d^{-1}\big)\,, \qquad \Lambda_{21}^{ik,j}(d) = O\big(d^{-1}\big)\,, \tag{2.5.40}$$

$$\Lambda_{22}^{ik,jl}(d) = \begin{cases} 2I_r+O\big(d^{-1}\big) & \text{if } i=j=k=l,\\ I_r+O\big(d^{-1}\big) & \text{if } i=j\neq k=l,\\ O\big(d^{-1}\big) & \text{otherwise.}\end{cases} \tag{2.5.41}$$

Recall from section 2.5.3 that the matrix of $\operatorname{Var}(t_d,L_d)$ in this basis is $I_{r(n+1)}+O\big(d^{-1}\big)$. By Corollary 2.A.11, the distribution of (L_d,S_d) given $t_d=0$ is then a centered Gaussian whose variance matrix is, by blocks,

$$\widetilde{\Lambda}(d) = \left(\begin{array}{c|c|c} I_{nr} & & \\ \hline & 2I_{nr} & \\ \hline & & I_{\frac{rn(n-1)}{2}} \end{array}\right) + O\big(d^{-1}\big)\,, \tag{2.5.42}$$

where the empty blocks are zeros. Let Λ denote the leading term in equation (2.5.42) and let $(L,S)\sim\mathcal{N}(0,\Lambda)$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(T_x^*\mathbb{R}\mathcal{X}\oplus\operatorname{Sym}(T_x^*\mathbb{R}\mathcal{X}))$. By Lemma 2.A.12, $(\widetilde{L}_d,\widetilde{S}_d)$ converges in distribution to (L,S).

Let $U\sim\mathcal{N}(0,\operatorname{Id})$ in $T_x\mathbb{R}\mathcal{X}$ be independent of all the other variables. By (2.4.9),

$$R_s(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\left\langle \nabla_x^{2,d}s\,,(\nabla_x^d s)^{\dagger *}(U)\right\rangle^{\wedge 2}\right]. \tag{2.5.43}$$

As in the case of Riemannian random waves (2.5.21), we have:

$$\begin{aligned}\mathbb{E}&\left[\left|\det{}^\perp\left(\nabla_x^d s\right)\right|\operatorname{Tr}(R_s(x)^{\wedge m})\,\middle|\, s(x)=0\right]\\ &= \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}}\mathbb{E}\left[\left|\det{}^\perp\left(\widetilde{L}_d\right)\right|\operatorname{Tr}\left(\left(R(x)+d\,\mathbb{E}_U\left[\left\langle \widetilde{S}_d\,,\widetilde{L}_d^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right].\end{aligned} \tag{2.5.44}$$

The proof of Lemma 2.5.6 adapts immediately to this setting, so that:

$$\begin{aligned}\mathbb{E}&\left[\left|\det{}^\perp\left(\widetilde{L}_d\right)\right|\operatorname{Tr}\left(\left(R(x)+d\,\mathbb{E}_U\left[\left\langle \widetilde{S}_d\,,\widetilde{L}_d^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right]\\ &= d^m\frac{m!}{(2m)!}\mathbb{E}\left[\left|\det{}^\perp(L)\right|\operatorname{Tr}\left(\left\langle S\,,L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right]\left(1+O\big(d^{-1}\big)\right),\end{aligned} \tag{2.5.45}$$

where the error term is uniform in $x\in M$.

Lemma 2.5.9. *Let $(L,S)\sim\mathcal{N}(0,\Lambda)$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(T_x^*\mathbb{R}\mathcal{X}\oplus\operatorname{Sym}(T_x^*\mathbb{R}\mathcal{X}))$ and let $U\in T_xM$ be a standard Gaussian vector independent of (L,S). We have:*

$$\mathbb{E}\left[\left|\det{}^\perp(L)\right|\operatorname{Tr}\left(\left\langle S\,,L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] = (-1)^m(2\pi)^{\frac{n}{2}}(2m)!\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi\operatorname{Vol}\left(\mathbb{S}^{n}\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}.$$

Once this lemma is proved, we get Theorem 2.1.4 immediately by (2.5.17), (2.5.38), (2.5.44), (2.5.45) and Lemma 2.5.9. We sketch the proof of Lemma 2.5.9 which is, unsurprisingly, adapted from the proof of Lemma 2.5.7.

Proof. The only difference between Lemmas 2.5.7 and 2.5.9 comes from the definition of Λ which is not the same in the algebraic case. The proof is exactly the same as the proof of Lemma 2.5.7 until the definition of K (2.5.34). The $\Lambda^{ik,jl}$ are now given by (2.5.42), hence we have to change the definition of K. In this setting,

$$K(p,\sigma,\sigma') = \begin{cases} 1 & \text{if } \sigma(2p-1) = \sigma'(2p) \text{ and } \sigma(2p) = \sigma'(2p-1), \\ 0 & \text{otherwise,} \end{cases}$$

so that $\displaystyle\prod_{p=1}^{m} K(p,\mathrm{id},\tau)$ is 0, unless $\tau = \tau_0 = \displaystyle\prod_{p=1}^{m}((2p-1)(2p))$. Then (2.5.36) becomes:

$$\sum_{\tau\in\mathfrak{S}_{2m}} \varepsilon(\tau) \prod_{p=1}^{m} K(p,\mathrm{id},\tau) = \varepsilon(\tau_0) \prod_{p=1}^{m} K(p,\mathrm{id},\tau_0) = (-1)^m.$$

This explains why the factor $\left(-\frac{n+4}{n+2}\right)^m$ becomes $(-1)^m$ in the algebraic case. What remains of the proof is as in Lemma 2.5.7. $\square$

2.6 Two special cases

In some special cases, the covariance kernel is known explicitly. It is then possible to prove more precise results. In this section, we sketch what happens on the flat torus and in the real projective space. In these cases, we get the expectation of the volume and the Euler characteristic of our random submanifolds for fixed λ (resp. fixed d).

2.6.1 The flat torus

Let $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$ denote the torus of dimension n that we equip with the quotient of the Euclidean metric on $\mathbb{R}^n$. We have $\mathrm{Vol}(\mathbb{T}^n) = (2\pi)^n$. We identify functions on $\mathbb{T}^n$ and $(2\pi\mathbb{Z})^n$-periodic functions on $\mathbb{R}^n$. Then, the Laplacian is $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and it is known that its eigenvalues are the integers of the form $\|p\|^2 = \sum(p_i)^2$, with $p = (p_1,\dots,p_n) \in \mathbb{N}^n$. The eigenspace associated to 0 is spanned by the constant function $x \mapsto (2\pi)^{-\frac{n}{2}}$. For $\lambda > 0$, the eigenspace associated to λ is spanned by the normalized functions of the form:

$$x \mapsto \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \sin(\langle p\,, x\rangle) \qquad \text{and} \qquad x \mapsto \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \cos(\langle p\,, x\rangle),$$

where $p \in \mathbb{Z}^n$ is such that $\|p\|^2 = \lambda$, and $\langle \cdot\,, \cdot\rangle$ is the canonical scalar product on $\mathbb{R}^n$.

We set $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\|^2 \leqslant \lambda\}$. After some computations, we get e_λ, the spectral function of the Laplacian on $\mathbb{T}^n$:

$$\forall \lambda \geqslant 0,\ \forall x, y \in \mathbb{T}^n, \qquad e_\lambda(x,y) = \frac{1}{(2\pi)^n} \sum_{p\in\mathbb{B}_\lambda} \cos(\langle p\,, x-y\rangle). \tag{2.6.1}$$

Let $\lambda \geqslant 0$ and $r \in \{1,\dots,n\}$, let V_λ be spanned by the eigenfunctions of Δ associated to eigenvalues smaller than λ, and let $f \sim \mathcal{N}(0,\mathrm{Id})$ in $(V_\lambda)^r$. For all $x \in \mathbb{T}^n$, $j^2_x(f)$ is a centered Gaussian variable whose variance is determined by Lemma 2.2.4 , Lemma 2.2.6

and the above formula (2.6.1). Note that e_λ and its derivatives are constant along the diagonal.

We can compute explicitly the variance of $j^2_x(f)$ (which is independent of x) and follow the same steps as in sections 2.5.2 and 2.5.4. Only, this time, we can make exact computations with λ fixed, instead of deriving asymptotics. These computations are not difficult, and similar to what we already did, so we simply state the final results. Note that the scaling of the variables has to be adapted.

Proposition 2.6.1. *On the flat torus $\mathbb{T}^n$, let $\lambda \geqslant 0$ and let $f_1, \dots, f_r$ be independent standard Gaussian functions in V_λ, with $1 \leqslant r \leqslant n$. We have:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1,\dots,p_n)\in\mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{r}{2}} (2\pi)^n \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\|^2 \leqslant \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of $\mathbb{B}_\lambda$.

This result was already known, see [RW08], where Rudnick and Wigman compute the variance of $\mathrm{Vol}\,(Z_f)$ when $r = 1$, and the references therein.

Proposition 2.6.2. *On the flat torus $\mathbb{T}^n$, let $\lambda \geqslant 0$ and let $f_1, \dots, f_r$ be independent standard Gaussian functions in V_λ, with $1 \leqslant r \leqslant n$. If $n - r$ is even, we have:*

$$\mathbb{E}[\chi(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1,\dots,p_n)\in\mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{n}{2}} (2\pi)^n \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right)\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi\,\mathrm{Vol}\,(\mathbb{S}^n)\,\mathrm{Vol}\,(\mathbb{S}^{n-1})},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\|^2 \leqslant \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of $\mathbb{B}_\lambda$.

Remark 2.6.3. For this last result, one of the points that make the computations tractable is that the Riemann tensor of the ambient manifold is zero.

2.6.2 The projective space

We consider the algebraic case with $\mathcal{X} = \mathbb{CP}^n$, $\mathcal{E} = \mathbb{C}^r \times \mathbb{CP}^n$ with the standard Hermitian metric on each fiber, and $\mathcal{L} = \mathcal{O}(1)$ the hyperplane bundle with its usual metric. Then, ω is the standard Fubini-Study form. We consider the real structures induced by the standard conjugation in $\mathbb{C}$.

Let e_d denote the Bergman kernel of $\mathcal{L}^d$ and E_d denote the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$. Since $\mathcal{E}$ is trivial we have a product situation, as in section 2.2.5, and E_d and e_d are related as in Lemma 2.2.6. Let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{C}^r$, for all x, $y \in \mathbb{CP}^n$,

$$E_d(x, y) = \left(\sum_{q=1}^{r} \zeta_q \otimes \zeta_q \right) \otimes e_d(x, y). \qquad (2.6.2)$$

In this case, $\mathbb{R}\mathcal{X} = \mathbb{RP}^n$ and $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ equals $\mathbb{R}^{hom}_d[X_0, \dots, X_n]$, the space of real homogeneous polynomials of degree d in $n+1$ variables. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, then we set $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$, $|\alpha| = \alpha_0 + \cdots + \alpha_n$ and, if $|\alpha| = d$, we also set $\binom{d}{\alpha} = \frac{d!}{\alpha_0! \cdots \alpha_n!}$.

It is well-known that an orthonormal basis of $\mathbb{R}^{hom}_d[X_0, \dots, X_n]$ for the inner product (2.2.16) is given by the sections:

$$s_\alpha = \sqrt{\frac{(n+d)!}{\pi^n d!}\binom{d}{\alpha}}\, X^\alpha, \qquad \text{with } |\alpha| = d, \qquad (2.6.3)$$

see [BSZ00, BBL96, Bü06, Kos93]. Then, formally,

$$e_d = \frac{(n+d)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} X^\alpha Y^\alpha = \frac{(n+d)!}{\pi^n d!} \left\langle X \,, Y \right\rangle^d . \tag{2.6.4}$$

More precisely, we consider the local coordinates $(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$, defined on a neighborhood of $[1 : 0 : \dots : 0]$, and the real holomorphic frame $s_{(d,0,\dots,0)}$ for $\mathcal{O}(d)$ on this neighborhood. In these coordinates,

$$e_d(x,y) = (1 + \langle x \,, y \rangle)^d \left(s_{(d,0,\dots,0)}(x) \otimes s_{(d,0,\dots,0)}(y) \right), \tag{2.6.5}$$

with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Since everything is invariant under orthogonal transformations of $\mathbb{RP}^n$, this totally describes e_d. In particular, the covariances that appear in the proofs of Theorems 2.1.3 and 2.1.4 are given, in this case, by the values of the function

$$(x,y) \mapsto \frac{(n+d)!}{\pi^n d!} (1 + \langle x \,, y \rangle)^d$$

and its derivatives at the point $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$.

Let $d \in \mathbb{N}$, $r \in \{1, \dots, n\}$ and $s \sim \mathcal{N}(0, \mathrm{Id})$ in $(\mathbb{R}_d^{hom}[X_0, \dots, X_n])^r$. For every $x \in \mathbb{RP}^n$, $j_x^{2,d}(s)$ is a centered Gaussian variable whose variance is determined by Lemma 2.2.13, (2.6.2) and (2.6.5). Once again, we can compute the variance of $j_x^{2,d}(s)$ explicitly, and it does not depend on $x \in \mathbb{RP}^n$. Then, we can follow the steps of the proof of Theorem 2.1.3, and make exact computations for fixed degree d. This yields Kostlan's result [Kos93]. Note that we have to adapt the scaling of the variables.

Theorem 2.6.4 (Kostlan)**.** *In the real projective space $\mathbb{RP}^n$, let $d \in \mathbb{N}$ and let $P_1, \dots, P_r$ be independent standard Gaussian polynomials in $\mathbb{R}_d^{hom}[X_0, \dots, X_n]$, with $1 \leqslant r \leqslant n$. Let Z_s denote the common zero set of $P_1, \dots, P_r$. We have:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_s)] = d^{\frac{r}{2}}\, \mathrm{Vol}\left(\mathbb{RP}^{n-r}\right).$$

Assuming that $n-r$ is even and equals $2m$, we can also adapt the proof of Theorem 2.1.4 to get the expected Euler characteristic for fixed d. This computation is a bit more complex than on the torus since the Riemann curvature of $\mathbb{RP}^n$ is not zero.

By Kac–Rice formula and equations (2.6.2) and (2.4.9),

$$\begin{aligned} \mathbb{E}[\chi(Z_s)] = \frac{1}{m!(2\pi)^{\frac{n}{2}}} \int_{x \in \mathbb{RP}^n} \frac{1}{(e_d(x,x))^{\frac{n}{2}}} \times \cdots \\ \mathbb{E}\left[\left| \det{}^\perp\left(\nabla_x^d s\right) \right| \mathrm{Tr}\left(\left(R(x) + \frac{1}{2} \mathbb{E}\left[\left\langle \nabla_x^{2,d} s \,, (\nabla_x^d s)^{\dagger *}(U) \right\rangle^{\wedge 2} \right] \right)^{\wedge m} \right) \right] \left| \mathrm{d}V_{\mathbb{RP}^n} \right|, \end{aligned} \tag{2.6.6}$$

where R is the Riemann curvature of $\mathbb{RP}^n$ and $U \sim \mathcal{N}(0, \mathrm{Id})$ in $T_x\mathbb{RP}^n$. We used the fact that $s(x)$, $\nabla_x^d s$ and $\nabla_x^{2,d} s$ are independent in this particular case. This is why the expectation is not conditioned on $s(x) = 0$.

Since everything is invariant under orthogonal transformations of $\mathbb{RP}^n$, we only need to compute the expectation in the integrand at the point $x = [1 : 0 : \dots : 0]$. We do this in the same chart as for (2.6.5) above. We get:

$$R(x) = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} dx_i \wedge dx_j \otimes dx_i \wedge dx_j.$$

Following the computations of section 2.5.4, we get (after a suitable rescaling) that the expectation in the integrand of (2.6.6) equals:

$$d^{\frac{r}{2}} \sum_{q=0}^{m} \frac{(d-1)^q}{2^q} \binom{m}{q} \mathbb{E}\left[\frac{\|X_{n-r+2}\| \dots \|X_n\|}{\|X_{n-r+1}\|^{2q-1}}\right] \mathbb{E}\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S \,, U' \right\rangle^{\wedge 2q}\right)\right], \quad (2.6.7)$$

where $U' \in \mathbb{R}^r$, $S \in (\mathbb{R}\mathcal{O}(d)_x)^r \otimes \operatorname{Sym}(T_x^*\mathbb{RP}^n)$ and $X_p \in \mathbb{R}^p$ (for $n - r + 1 \leqslant p \leqslant n$) are globally independent standard Gaussian vectors. By the same kind of computations as in the proof of Lemma 2.5.7, we get:

$$\mathbb{E}_S\left[\left\langle S \,, U' \right\rangle^{\wedge 2}\right] = -\left\|U'\right\|^2 \sum_{1 \leqslant i,j \leqslant n} dx_i \wedge dx_j \otimes dx_i \wedge dx_j. \quad (2.6.8)$$

Restricting R and S to the span of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}\right)$, we have:

$$\forall q \in \{0, \dots, m\}, \qquad \operatorname{Tr}\left(\mathbb{R}(x)^{\wedge m-q} \wedge \mathbb{E}_S\left[\left\langle S \,, U' \right\rangle^{\wedge 2}\right]^{\wedge q}\right) = (-1)^q \frac{1}{2^{m-q}} (2m)! \left\|U'\right\|^{2q}.$$

As in the proof of Lemma 2.5.7, we compute the expectation of this term with respect to the variable U' by Lemma 2.A.13. The same lemma allows us to compute the value of (2.6.7). Finally, we recover Bürgisser's result [Bü06].

Theorem 2.6.5 (Bürgisser)**.** *In the real projective space $\mathbb{RP}^n$, let $d \in \mathbb{N}$ and let $P_1, \dots, P_r$ be independent standard Gaussian polynomials in $\mathbb{R}_d^{hom}[X_0, \dots, X_n]$, with $1 \leqslant r \leqslant n$. Let Z_s denote the common zero set of $P_1, \dots, P_r$. If $n - r$ is even, we have:*

$$\mathbb{E}[\chi(Z_s)] = d^{\frac{r}{2}} \sum_{p=0}^{\frac{n-r}{2}} (1-d)^p \, \frac{\Gamma\left(p + \frac{r}{2}\right)}{p! \; \Gamma\left(\frac{r}{2}\right)}.$$

2.A Concerning Gaussian vectors

In this appendix we survey the few facts we need concerning random vectors, especially Gaussian ones. It is essentially borrowed from [Nic15c, Appendix A]. We include it here for the reader's convenience.

2.A.1 Variance and covariance as tensors

Let V be a real vector space of finite dimension and X a random vector with values in V. For any $\xi \in V^*$, $\xi(X)$ is a real random variable. From now on, we assume that these variables are square integrable.

Definition 2.A.1. The *expectation* (or *mean*) of X is the linear form on V^* defined by:

$$m_X : \xi \mapsto \mathbb{E}[\xi(X)]. \quad (2.A.1)$$

If $m_X = 0$, we say that X (resp. its distribution $\mathrm{d}P_X$) is *centered.*

Under the canonical isomorphism $V^{**} \simeq V$, we have $m_X = \displaystyle\int_V x \; \mathrm{d}P_X$.

Definition 2.A.2. The *variance* of X is the non-negative symmetric bilinear form on V^* defined by:

$$\operatorname{Var}(X) : (\xi, \eta) \mapsto \mathbb{E}[\xi(X - m_X)\eta(X - m_X)]. \quad (2.A.2)$$

Remark 2.A.3. Traditionally, the term "variance" is only used when V has dimension 1 and one speaks of "covariance" when $\dim(V) \geqslant 2$. We chose to use the term "covariance" for couples of distinct random vectors (see below) and "variance" otherwise. This is the convention of [AW09], for example.

As a bilinear form on $V^* \times V^*$, $\mathrm{Var}(X)$ is naturally an element of $V \otimes V$ and we have the following lemma.

Lemma 2.A.4. *Let X be a random vector in V, then we have:*

$$\mathrm{Var}(X) = \mathbb{E}[(X - m_X) \otimes (X - m_X)]\,. \tag{2.A.3}$$

Proof. For any ξ and $\eta \in V^*$, we have:

$$\begin{aligned}\mathrm{Var}(X)\,(\xi,\eta) &= \mathbb{E}[\xi(X - m_X)\eta(X - m_X)] = \mathbb{E}[(\xi \otimes \eta)((X - m_X) \otimes (X - m_X))] \\ &= (\xi \otimes \eta)\mathbb{E}[(X - m_X) \otimes (X - m_X)]\,.\end{aligned}$$

□

Definition 2.A.5. The *variance operator* of X is the linear map $\Lambda_X : V^* \to V$ such that, for any ξ and $\eta \in V^*$,

$$\xi\,(\Lambda_X \eta) = \mathrm{Var}(X)\,(\xi,\eta). \tag{2.A.4}$$

By Lemma 2.A.4 we have:

$$\Lambda_X : \eta \mapsto \mathbb{E}[(X - m_X) \otimes \eta(X - m_X)]\,. \tag{2.A.5}$$

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2)$, with X_i a random vector in V_i, then $m_X = m_{X_1} + m_{X_2}$ and the variance form $\mathrm{Var}(X)$ splits accordingly into four parts:

$$\begin{aligned}&\mathrm{Var}(X_1) : V_1^* \times V_1^* \to \mathbb{R}, && && \mathrm{Cov}(X_1, X_2) : V_1^* \times V_2^* \to \mathbb{R},\\ &\mathrm{Var}(X_2) : V_2^* \times V_2^* \to \mathbb{R} && \text{and} && \mathrm{Cov}(X_2, X_1) : V_2^* \times V_1^* \to \mathbb{R}.\end{aligned}$$

These bilinear forms are associated, as above, to the following operators:

$$\begin{aligned}&\Lambda_{11} : V_1^* \to V_1, && && \Lambda_{12} : V_2^* \to V_1,\\ &\Lambda_{22} : V_2^* \to V_2 && \text{and} && \Lambda_{21} : V_1^* \to V_2.\end{aligned}$$

Since $\mathrm{Var}(X)$ is symmetric, $\mathrm{Cov}(X_1, X_2)\,(\xi,\eta) = \mathrm{Cov}(X_2, X_1)\,(\eta,\xi)$ for any ξ and η.

Definition 2.A.6. We say that $\mathrm{Cov}(X_1, X_2)$ is the *covariance* of X_1 and X_2, and that Λ_{12} is their *covariance operator*.

As above, $\mathrm{Cov}(X_1, X_2)$ is naturally an element of $V_1 \otimes V_2$.

Lemma 2.A.7. *Let X_1 and X_2 be random vectors in V_1 and V_2 respectively, then we have:*

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})]\,.$$

Moreover, for any $\eta \in V_2^*$, $\qquad \Lambda_{12}(\eta) = \mathbb{E}[(X_1 - m_{X_1}) \otimes \eta(X_2 - m_{X_2})]$.

Let $L : V \to V'$ be a linear map between finite-dimensional vector spaces and X be a random vector in V. Then $L(X)$ is a random vector in V' with $\mathrm{d}P_{L(X)} = L_*(\mathrm{d}P_X)$. An immediate consequence of (2.A.1), (2.A.2) and (2.A.4) is that:

$$m_{L(X)} = m_X \circ L^*, \tag{2.A.6}$$

$$\mathrm{Var}(L(X)) = \mathrm{Var}(X)\,(L^*\cdot, L^*\cdot), \tag{2.A.7}$$

and

$$\Lambda_{L(X)} = L\Lambda_X L^*, \tag{2.A.8}$$

where $L^* : (V')^* \to V^*$ is defined by $L^* : \xi \mapsto (\xi \circ L)$.

If X is a random vector in a Euclidean space $(V, \langle \cdot \,, \cdot \rangle)$, we can see $\mathrm{Var}(X)$ as a bilinear symmetric form on V, and Λ_X as a self-adjoint operator on V. Then, by (2.A.3) and (2.A.5) :

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}[(X - m_X)^* \otimes (X - m_X)^*] \,, \\ \Lambda_X &= \mathbb{E}[(X - m_X) \otimes (X - m_X)^*] \,, \end{aligned}$$

where for any $v \in V$, we set $v^* = \langle v \,, \cdot \rangle \in V^*$.

If $V = V_1 \oplus V_2$, we can see Λ_{12} as a linear operator from V_2 to V_1 and by Lemma 2.A.7:

$$\begin{aligned} \mathrm{Cov}(X_1, X_2) &= \mathbb{E}[(X_1 - m_{X_1})^* \otimes (X_2 - m_{X_2})^*] \,, \\ \Lambda_{12} &= \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})^*] \,, \end{aligned}$$

Lemma 2.A.8. *Let X be a random vector in a Euclidean space V, then we have:*

$$\forall v \in V, \forall w \in V, \quad \mathbb{E}[\langle v \,, X - m_X \rangle \langle w \,, X - m_X \rangle] = \langle v \,, \Lambda_X w \rangle \,. \tag{2.A.9}$$

Proof. Let v and $w \in V$, then we have:

$$\begin{aligned} \mathbb{E}[\langle v \,, X - m_X \rangle \langle w \,, X - m_X \rangle] &= \mathrm{Var}(X)\,(v, w) && \text{as a bilinear form on } V, \\ &= \langle v \,, \Lambda_X w \rangle && \text{where } \Lambda_X : V \to V. \end{aligned}$$ ☐

2.A.2 Gaussian vectors

The following material can be found either in [AW09, section 1.2] or [TA07, section 1.2]. We present it in a coordinate-free fashion, in the spirit of [Nic15c].

Let $m \in \mathbb{R}$ and $\sigma \geqslant 0$, then the *Gaussian* (or *normal*) distribution on $\mathbb{R}$ with expectation m and variance σ^2 is the distribution whose characteristic function is

$$\xi \longmapsto \exp\left(im\xi - \frac{1}{2}\sigma^2\xi^2\right).$$

If $\sigma = 0$, this is the Dirac measure centered at m, otherwise it has a density with respect to the Lebesgue measure, given by $x \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$.

Let V be a real vector space of dimension n, then a random vector X in V is said to be *Gaussian*, or *normally distributed*, if for any $\xi \in V^*$, $\xi(X)$ is a Gaussian variable in $\mathbb{R}$. Recall that a Gaussian vector has finite moments of all orders and that its distribution is totally determined by its expectation and variance. We denote by $\mathcal{N}(m, \Lambda)$ the Gaussian distribution with expectation m and variance operator Λ and by $X \sim \mathcal{N}(m, \Lambda)$ the fact that X is distributed according to $\mathcal{N}(m, \Lambda)$. From (2.A.6) and (2.A.8) we deduce the following.

Lemma 2.A.9. *Let $L : V \to V'$ be a linear map between finite-dimensional vector spaces and $X \sim \mathcal{N}(m, \Lambda)$ in V. Then $L(X) \sim \mathcal{N}(Lm, L\Lambda L^*)$ in V'.*

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2) \sim \mathcal{N}(m, \Lambda)$ then, with the notations of section 2.A.1, Lemma 2.A.9 shows that $X_1 \sim \mathcal{N}(m_{X_1}, \Lambda_{11})$ and $X_2 \sim \mathcal{N}(m_{X_2}, \Lambda_{22})$. Besides, X_1 and X_2 are independent if and only if $\mathrm{Cov}(X_1, X_2) = 0$, or equivalently $\Lambda_{12} = 0$.

Proposition 2.A.10 (Regression formula)**.** *Let $X = (X_1, X_2)$ be a Gaussian vector in $V_1 \oplus V_2$. If $\mathrm{Var}(X_1)$ is non-degenerate then X_2 has the same distribution as*

$$m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}(X_1 - m_{X_1}) + Y$$

where Y is a centered Gaussian vector in V_2 with variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$, independent of X_1.

This is shown in [AW09, prop. 1.2]. From this, we deduce that the distribution of X_2 given $X_1 = x_1$ is Gaussian in V_2 with expectation $m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}(x_1 - m_{X_1})$ and variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$. We use this in the case where X is centered and $x_1 = 0$.

Corollary 2.A.11. *Let $X = (X_1, X_2)$ be a centered Gaussian vector in $V_1 \oplus V_2$ and assume that $\mathrm{Var}(X_1)$ is non-degenerate. Then the distribution of X_2 given $X_1 = 0$ is a centered Gaussian in V_2 with variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$.*

In what follows, we assume that V is a Euclidean space. Recall that in this case, we can see the variance operator of a random vector as an endomorphism of V. We will say that $\mathcal{N}(0, \mathrm{Id})$ is the *standard normal distribution* on V, where Id denotes the identity map on V.

If $\mathrm{Var}(X)$ is non-degenerate, then $\mathrm{d}P_X$ has the following density with respect to the Lebesgue measure on V:

$$x \longmapsto \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Lambda_X)}} \exp\left(-\frac{1}{2}\left\langle (\Lambda_X)^{-1}(x-m), x-m\right\rangle\right).$$

If $\mathrm{Var}(X)$ is singular, $\mathrm{d}P_X$ is supported on $\ker(\Lambda_X)^{\perp}$.

Lemma 2.A.12. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of random vectors in V such that, for all $k \in \mathbb{N}$, $X_k \sim \mathcal{N}(m_k, \Lambda_k)$. We assume that $m_k \xrightarrow[k\to+\infty]{} m$ and $\Lambda_k \xrightarrow[k\to+\infty]{} \Lambda$. Then X_k converges in distribution to $\mathcal{N}(m, \Lambda)$.*

Proof. Under the hypothesis of the lemma, the characteristic function of X_k converges pointwise to the characteristic function of a Gaussian vector $X \sim \mathcal{N}(m, \Lambda)$. Then, Lévy's continuity theorem gives the result. □

We conclude this appendix by computing two Gaussian expectations.

Lemma 2.A.13. *Let $X \sim \mathcal{N}(0, \mathrm{Id})$ with values in a Euclidean space of dimension n and let $k \in \mathbb{Z}$ such that $k > -n$. Then,*

$$\mathbb{E}\left[\|X\|^k\right] = (2\pi)^{\frac{k}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n+k-1}\right)}.$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\|X\|^k\right] &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_V \|x\|^k e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} r^{k+n-1}e^{-\frac{1}{2}r^2}\,\mathrm{d}r \\
&= \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} (2t)^{\frac{k+n}{2}-1}e^{-t}\,\mathrm{d}t = \frac{(2\pi)^{\frac{k}{2}}\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{2\pi^{\frac{k+n}{2}}}\,\Gamma\left(\frac{k+n}{2}\right).
\end{aligned}$$ □

Assuming we proved Proposition 2.5.8 (more precisely its Corollary 2.B.3), we have the following.

Lemma 2.A.14. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. Let $L \sim \mathcal{N}(0, \mathrm{Id})$ in $V' \otimes V^*$. Then:*

$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right|\right] = (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}.$$

Proof. By Corollary 2.B.3, $\left|\det^{\perp}(L)\right|$ is distributed as $\|X_{n-r+1}\| \cdots \|X_n\|$, with X_p a standard Gaussian in $\mathbb{R}^p$ for all p and $X_{n-r+1}, \dots, X_n$ independent. Then, using Lemma 2.A.13,

$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right|\right] = \mathbb{E}[\|X_{n-r+1}\| \cdots \|X_n\|] = \prod_{p=n-r+1}^{n} \mathbb{E}[\|X_p\|] = (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}. \qquad \square$$

2.B Proof of Proposition 2.5.8

This appendix is devoted to the proof of Proposition 2.5.8 which is a reformulation of [Bü06, prop. 3.12]. Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. The space $V' \otimes V^*$ of linear maps from V to V' comes with a natural scalar product induced by those on V and V'. The set of linear maps of rank less than r is an algebraic submanifold of $V' \otimes V^*$ of codimension at least 1, hence it has measure 0 for any non-singular Gaussian measure. Let L be a standard Gaussian vector in $V' \otimes V^*$, then the rank of L is r almost surely. Hence $L^\dagger$ is well-defined almost surely (recall Definition 2.4.7).

We introduce some further notations. Let $\mathcal{B} \subset V' \otimes V^*$ denote the set of maps of rank r. We set $\mathcal{F} = \{(L, U) \in \mathcal{B} \times V \mid U \in \ker(L)^{\perp}\}$ and $\mathcal{S} = \{(L, U) \in \mathcal{B} \times V \mid U \in \mathbb{S}(\ker(L)^{\perp})\}$. Here and in the sequel, $\mathbb{S}(\cdot)$ stands for the unit sphere of the concerned space. Given $L \in \mathcal{B}$ and $U \in V$, we denote by $\widetilde{U}$ the orthogonal projection of U onto $\ker(L)^{\perp}$. Then we set:

$$\rho = \|\widetilde{U}\|, \qquad \theta = \frac{\widetilde{U}}{\|\widetilde{U}\|} \qquad \text{and} \qquad \theta' = \frac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|}. \tag{2.B.1}$$

Note that $L^*\theta' = \dfrac{(L^\dagger L)^*\theta}{\|(L^\dagger)^*\theta\|} = \dfrac{\theta}{\|(L^\dagger)^*\theta\|}$, hence $\|L^*\theta'\| = \dfrac{1}{\|(L^\dagger)^*\theta\|}$ and finally:

$$\theta = \frac{L^*\theta'}{\|L^*\theta'\|} \qquad \text{and} \qquad L^{\dagger *}U = L^{\dagger *}\widetilde{U} = \rho L^{\dagger *}\theta = \frac{\rho}{\|L^*\theta'\|}\theta'. \tag{2.B.2}$$

We choose orthonormal bases $(e_1, \dots, e_n)$ and $(e'_1, \dots, e'_r)$, of V and V' respectively, such that $e_r = \theta$, $e'_r = \theta'$ and $(e_1, \dots, e_r)$ is a basis of $\ker(L)^{\perp}$. Then,

$$\forall i \in \{1, \dots, n\}, \quad \left\langle Le_i \,, \theta' \right\rangle = \left\langle e_i \,, L^*\theta' \right\rangle = \left\|L^*\theta'\right\| \left\langle e_i \,, \theta \right\rangle.$$

Thus the matrix of L in these bases has the form:

$$\left(\begin{array}{c|c|c} A & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \\ \hline 0 \cdots 0 & \|L^*\theta'\| & 0 \cdots 0 \end{array}\right), \tag{2.B.3}$$

and $\left|\det^{\perp}(L)\right| = |\det(A)| \, \|L^*\theta'\|$.

Let π_θ and $\pi_{\theta'}$ denote the orthogonal projections along $\mathbb{R} \cdot \theta$ in V and along $\mathbb{R} \cdot \theta'$ in V' respectively. We define $L' : V \to (\mathbb{R} \cdot \theta')^{\perp}$ by $L' = \pi_{\theta'} \circ L \circ \pi_\theta$. Then $|\det(A)| = \left|\det^{\perp}(L')\right|$, and L' does not depend on our choice of bases. Finally, we have:

$$\left(\left|\det{}^{\perp}(L)\right|, L^{\dagger *}U\right) = \left(\left|\det{}^{\perp}(L')\right| \left\|L^*\theta'\right\|, \frac{\rho\theta'}{\|L^*\theta'\|}\right). \tag{2.B.4}$$

To prove Proposition 2.5.8, we will show that $\left|\det^{\perp}(L')\right|$, $\|L^*\theta'\|$, ρ and θ' are independent and identify their distributions.

If L and U are independent standard Gaussians, then almost surely $L \in \mathcal{B}$ and we can consider (L, U) as a random element of $\mathcal{B} \times V$. Then $(L, \widetilde{U})$ is a random element of $\mathcal{F}$ and its distribution is characterized by:

$$\begin{aligned}\mathbb{E}\left[\phi(L,\widetilde{U})\right] &= \int_{L\in\mathcal{B}}\left(\int_{U\in V}\phi(L,\widetilde{U})\,\mathrm{d}\nu_n(U)\right)\mathrm{d}\nu_{nr}(L)\\ &= \int_{L\in\mathcal{B}}\left(\int_{\widetilde{U}\in\ker(L)^{\perp}}\phi(L,\widetilde{U})\,\mathrm{d}\nu_r(\widetilde{U})\right)\mathrm{d}\nu_{nr}(L),\end{aligned}$$

for any bounded continuous function $\phi : \mathcal{F} \to \mathbb{R}$. Recall that $\mathrm{d}\nu_N$ stands for the standard Gaussian measure in dimension N. We get the distribution of $(L, \theta, \rho) \in \mathcal{S} \times \mathbb{R}_+$ by a polar change of variables in the innermost integral: for any bounded continuous $\phi : \mathcal{S} \times \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}[\phi(L,\theta,\rho)] = \int_{L\in\mathcal{B}}\int_{\theta\in\mathbb{S}(\ker(L)^{\perp})}\int_{\rho=0}^{+\infty}\phi(L,\theta,\rho)\rho^{r-1}e^{-\frac{\rho^2}{2}}\frac{\mathrm{d}\rho}{(2\pi)^{\frac{r}{2}}}\,\mathrm{d}\theta\,\mathrm{d}\nu_{nr}(L),$$

where $\mathrm{d}\rho$ is the Lebesgue measure on $\mathbb{R}$ and $\mathrm{d}\theta$ is the Euclidean measure on the sphere $\mathbb{S}(\ker(L)^{\perp})$.

This distribution is a product measure on $\mathcal{S} \times \mathbb{R}_+$, thus (L, θ) and ρ are independent variables. Since $\left(\left|\det^{\perp}(L')\right|, \theta', \|L^*\theta'\|\right)$ only depends on (L, θ), this triple is independent of ρ. Besides, ρ is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^r$ since its density with respect to the Lebesgue measure is $\rho \mapsto \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)(2\pi)^{-\frac{r}{2}}\rho^{r-1}e^{-\frac{\rho^2}{2}}$ on $\mathbb{R}_+$ and vanishes elsewhere. Finally, the distribution of (L, θ) satisfies:

$$\mathbb{E}[\phi(L,\theta)] = \int_{L\in\mathcal{B}}\int_{\theta\in\mathbb{S}(\ker(L)^{\perp})}\phi(L,\theta)\frac{\mathrm{d}\theta}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\,\mathrm{d}\nu_{nr}(L), \tag{2.B.5}$$

for any bounded and continuous $\phi : \mathcal{S} \to \mathbb{R}$.

We will now compute the distribution of (L, θ') in $\mathcal{B} \times \mathbb{S}(V')$.

Lemma 2.B.1. *For any bounded and continuous $\phi : \mathcal{B} \times \mathbb{S}(V') \to \mathbb{R}$,*

$$\mathbb{E}\left[\phi(L,\theta')\right] = \int_{\theta'\in\mathbb{S}(V')}\int_{L\in\mathcal{B}}\phi(L,\theta')\frac{\left|\det^{\perp}(L')\right|}{\|L^*\theta'\|^{r-1}}e^{-\frac{\|L\|^2}{2}}\frac{\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\frac{\mathrm{d}L}{(2\pi)^{\frac{nr}{2}}}, \tag{2.B.6}$$

where $\mathrm{d}\theta'$ is the Euclidean measure on $\mathbb{S}(V')$, $\mathrm{d}L$ is the Lebesgue measure on $V' \otimes V^*$ and L' is defined as in (2.B.4).

Proof. Fixing some ϕ, we see from (2.B.5) and (2.B.1) that:

$$\mathbb{E}\left[\phi(L,\theta')\right] = \int_{L\in\mathcal{B}}\int_{\theta\in\mathbb{S}(\ker(L)^{\perp})}\phi\left(L, \frac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|}\right)e^{-\frac{\|L\|^2}{2}}\frac{\mathrm{d}\theta}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\frac{\mathrm{d}L}{(2\pi)^{\frac{nr}{2}}}.$$

Then we make the change of variables $\theta' = \psi(\theta) = \dfrac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|}$ in the innermost integral, with L fixed. Recalling (2.B.2), we have $\psi^{-1} : \theta' \mapsto \dfrac{L^*\theta'}{\|L^*\theta'\|}$ from $\mathbb{S}(V')$ to $\mathbb{S}(\ker(L)^{\perp})$. Now, the differential of ψ^{-1} at $\theta' \in \mathbb{S}(V')$ satisfies:

$$\forall v \in (\mathbb{R}\cdot\theta')^{\perp}, \quad d_{\theta'}(\psi^{-1})\cdot v = \frac{1}{\|L^*\theta'\|}\left(L^*v - \left\langle L^*v\,, \frac{L^*\theta'}{\|L^*\theta'\|}\right\rangle\frac{L^*\theta'}{\|L^*\theta'\|}\right) = \frac{\pi_\theta(L^*v)}{\|L^*\theta'\|}.$$

As above, we choose an orthonormal basis $\left(e'_1, \ldots, e'_{r-1}\right)$ of $(\mathbb{R} \cdot \theta')^\perp$ and an orthonormal basis $(e_1, \ldots, e_{r-1})$ of $(\mathbb{R} \cdot \theta \oplus \ker(L))^\perp$. In these coordinates we have:

$$\left|\det(d_{\theta'}(\psi^{-1}))\right| = \frac{\left|\det\left(\pi_\theta \circ L^*_{/(\theta')^\perp}\right)\right|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A^*)|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A)|}{\|L^*\theta'\|^{r-1}} = \frac{\left|\det^\perp(L')\right|}{\|L^*\theta'\|^{r-1}},$$

where A is as in (2.B.3) and L' as in (2.B.4). This proves (2.B.6). $\square$

We can now compute the joint distribution of $\left(\left|\det^\perp(L')\right|, \theta', \|L^*\theta'\|\right)$ from the one of (L, θ'). We fix $\theta' \in \mathbb{S}(V')$ and an orthonormal basis $(e'_1, \ldots e'_r)$ of V' such that $e'_r = \theta'$. The choice of $L \in V' \otimes V^*$ is equivalent to the choice of the r independent standard Gaussian vectors $L^*e'_1, \ldots, L^*e'_r$ in V. For simplicity, we set $L_i = L^*e'_i$. Note that if we choose a basis for V as well, these are the rows of the matrix of L. We can rewrite (2.B.6) as:

$$\mathbb{E}\left[\phi(L,\theta')\right] = \int_{\theta'\in\mathbb{S}(V')} \int_{L_1,\ldots,L_{r-1}\in V} \int_{L_r\in V} \phi(L,\theta') \frac{\left|\det^\perp(L')\right|}{\|L_r\|^{r-1}} \frac{e^{-\frac{1}{2}\sum\|L_i\|^2}\,\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)} \frac{\mathrm{d}L_1\cdots\mathrm{d}L_r}{(2\pi)^{\frac{nr}{2}}},$$

where $\mathrm{d}L_i$ denotes the Lebesgue measure in the i-th copy of V. We set $\alpha_r = \frac{L_r}{\|L_r\|}$ and $\rho_r = \|L_r\|$. Here, $L' = \pi_{\theta'} \circ L \circ \pi_{\alpha_r}$ depends on α_r and θ' but not on ρ_r. Making a polar change of variables, the above integral equals:

$$\int_{\substack{\theta'\in\mathbb{S}(V')\\ \alpha_r\in\mathbb{S}(V)\\ L_1,\ldots,L_{r-1}}} \int_{\rho_r=0}^{+\infty} \phi(L,\theta') \frac{(\rho_r)^{n-r}\, e^{-\frac{\rho_r^2}{2}}\,\mathrm{d}\rho_r}{(2\pi)^{\frac{n-r+1}{2}}} \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\|L_i\|^2}\left|\det^\perp(L')\right|}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)} \frac{\mathrm{d}\theta'\,\mathrm{d}\alpha_r\,\mathrm{d}L_1\cdots\mathrm{d}L_{r-1}}{(2\pi)^{\frac{(n+1)(r-1)}{2}}}.$$

Then, $\rho_r = \|L^*\theta'\|$ is independent of $(\theta', \alpha_r, L_1, \ldots, L_{r-1})$, hence of $\left(\theta', \left|\det^\perp(L')\right|\right)$. Moreover ρ_r is distributed as the norm of a standard Gaussian in $\mathbb{R}^{n-r+1}$, since it has the same density. Finally, $(\theta', \alpha_r, L_1, \ldots, L_{r-1})$ has the density:

$$\left(\theta', \alpha_r, L_1, \ldots, L_{r-1}\right) \mapsto \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\|L_i\|^2}}{(2\pi)^{\frac{(n+1)(r-1)}{2}}} \frac{\left|\det^\perp(L')\right|}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}$$

with respect to $\mathrm{d}\theta' \otimes \mathrm{d}\alpha_r \otimes \mathrm{d}L_1 \otimes \cdots \otimes \mathrm{d}L_{r-1}$.

For $i \in \{1, \ldots, r-1\}$ we denote by $L_i^\perp$ the orthogonal projection of L_i onto the orthogonal of the subspace spanned by $(\alpha_r, L_1, \ldots, L_{i-1})$.

Lemma 2.B.2. *For any $L \in \mathcal{B}$, $\left|\det^\perp(L')\right| = \left\|L_1^\perp\right\| \cdots \left\|L_{r-1}^\perp\right\|$.*

Proof. If one of the $L_i^\perp$ is zero, then the vectors $\alpha_r, L_1, \ldots, L_{r-1}$ are linearly dependent and L is singular. Since we assumed $L \in \mathcal{B}$ this is not the case and $\left(\frac{L_1^\perp}{\|L_1^\perp\|}, \ldots, \frac{L_{r-1}^\perp}{\|L_{r-1}^\perp\|}\right)$ is an orthonormal basis of $\ker(L')^\perp$. Writing the matrix of the restriction of L' to $\ker(L')^\perp$ in this basis and $\left(e'_1, \ldots, e'_{r-1}\right)$, we see that it is lower triangular with diagonal coefficients $\left\|L_1^\perp\right\|, \ldots, \left\|L_{r-1}^\perp\right\|$. This proves the lemma. $\square$

Let ϕ be a continuous bounded function from $\mathbb{S}(V') \times \mathbb{R}_+$ to $\mathbb{R}$. We have:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(\theta', \left|\det{}^{\perp}(L')\right|\right)\right] \\
&\quad = \int \phi\left(\theta', \left|\det{}^{\perp}(L')\right|\right) \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\|L_i\|^2}\left|\det{}^{\perp}(L')\right|}{(2\pi)^{\frac{(n+1)(r-1)}{2}}} \frac{\mathrm{d}\theta' \,\mathrm{d}\alpha_r \,\mathrm{d}L_1 \dots \mathrm{d}L_{r-1}}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)} \\
&\quad = \int_{\alpha_r,\theta'} \int_{L_1^{\perp}} \cdots \int_{L_{r-1}^{\perp}} \phi\left(\theta', \prod_{i=1}^{r-1}\left\|L_i^{\perp}\right\|\right) \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{r-1}\left\|L_i^{\perp}\right\|^2\right)\prod_{i=1}^{r-1}\left\|L_i^{\perp}\right\|}{(2\pi)^{\frac{(2n+2-r)(r-1)}{4}}} \times \\
&\qquad\qquad \frac{\mathrm{d}L_{r-1}^{\perp}\dots \mathrm{d}L_1^{\perp}\,\mathrm{d}\theta' \,\mathrm{d}\alpha_r}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}.
\end{aligned}$$

Then we make polar changes of variables: for each i we set $\rho_i = \left\|L_i^{\perp}\right\|$ and $\alpha_i = \dfrac{L_i^{\perp}}{\left\|L_i^{\perp}\right\|}$. Note that, when $L_1, \dots, L_{i-1}$ are fixed, $L_i^{\perp}$ is a vector in a space of dimension $n-i$. We have:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(\theta', \left|\det{}^{\perp}(L')\right|\right)\right] \\
&\quad = \int \phi\left(\theta', \prod_{i=1}^{r-1}\rho_i\right) \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\rho_i^2}\prod_{i=1}^{r-1}(\rho_i)^{n-i}}{(2\pi)^{\frac{(2n+2-r)(r-1)}{4}}} \frac{\mathrm{d}\rho_1\dots\mathrm{d}\rho_{r-1}\,\mathrm{d}\alpha_1\dots\mathrm{d}\alpha_r\,\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)} \\
&\quad = \int_{\rho_1,\dots,\rho_{r-1},\theta'} \phi\left(\theta', \prod_{i=1}^{r-1}\rho_i\right) \prod_{i=1}^{r-1}\left(\mathrm{Vol}\left(\mathbb{S}^{n-i}\right)\frac{e^{-\frac{\rho_i^2}{2}}(\rho_i)^{n-i}}{(2\pi)^{\frac{n+1-i}{2}}}\right) \mathrm{d}\rho_1\dots\mathrm{d}\rho_{r-1}\frac{\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}.
\end{aligned}$$

This shows that $\theta', \rho_1, \dots, \rho_{r-1}$ are independent variables, that θ' is uniformly distributed in $\mathbb{S}(V')$ and that, for all $i \in \{1, \dots, r-1\}$, ρ_i is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^{n+1-i}$. Finally, this shows that $\left|\det^{\perp}(L')\right|$ is distributed as $\displaystyle\prod_{i=1}^{r-1}\rho_i$.

Putting all we have done so far together, we see that $\left(\left|\det^{\perp}(L)\right|, L^{\dagger *}U\right)$ is distributed as $\left(\|X_n\|\cdots\|X_{n-r+1}\|, \dfrac{\rho\theta'}{\|X_{n-r+1}\|}\right)$, where X_p is a standard Gaussian vector in $\mathbb{R}^p$ for all p. Moreover, θ' is uniformly distributed in $\mathbb{S}(V')$, ρ is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^r$, and all these variables are globally independent. Finally $U' = \rho\theta'$ is a standard Gaussian in V', independent of $X_n, \dots, X_{n-r+1}$ so we have proved Proposition 2.5.8. An immediate corollary of this is the following.

Corollary 2.B.3. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. Let $L \sim \mathcal{N}(0, \mathrm{Id})$ in $V' \otimes V^*$. Then $\left|\det^{\perp}(L)\right|$ is distributed as $\|X_{n-r+1}\|\cdots\|X_n\|$, where for all $p \in \{n-r+1, \dots, n\}$, $X_p \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^p$ and these vectors are globally independent.*

2.C Proof of the Kac–Rice formula

In this appendix, we give a proof of the Kac–Rice formula using Federer's coarea formula. This was already done by Bleher, Shiffman and Zelditch in [BSZ01, thm. 4.2]. See also [AW09, chap. 6].

2.C.1 The coarea formula

We start by stating the coarea formula in the case of a smooth map between smooth Riemannian manifolds. A proof in this special case can be found in [How93, Appendix] (see [Fed96, thm. 3.2.12] for the general case).

Let $\pi : \widetilde{M} \to M$ be a smooth map between smooth Riemannian manifolds of respective dimensions m and n. We assume that $m \geqslant n$. Let $\left|\mathrm{d}V_{\widetilde{M}}\right|$ (resp. $|\mathrm{d}V_M|$) denote the Riemannian measure on $\widetilde{M}$ (resp. M) induced by its metric. By Sard's theorem, for almost every $y \in M$, $\pi^{-1}(y)$ is a smooth submanifold of dimension $(m-n)$ of $\widetilde{M}$. For such $y \in M$, we denote by $|\mathrm{d}V_y|$ the Riemannian measure on $\pi^{-1}(y)$ induced by the metric of $\widetilde{M}$. When $m = n$, the dimension $\pi^{-1}(y)$ is 0 and $|\mathrm{d}V_y|$ is just $\displaystyle\sum_{x\in\pi^{-1}(y)} \delta_x$, where δ_x is the Dirac measure at x.

Theorem 2.C.1 (Coarea formula, Federer). *Let $\pi : \widetilde{M} \to M$ be a smooth map between smooth Riemannian manifolds of dimension m and n respectively. We assume that $m \geqslant n$. Let $\phi : \widetilde{M} \to \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{x\in\widetilde{M}} \phi(x) \left|\det{}^{\perp}(d_x\pi)\right| \left|\mathrm{d}V_{\widetilde{M}}\right| = \int_{y\in M} \left(\int_{x\in\pi^{-1}(y)} \phi(x) \,|\mathrm{d}V_y| \right) |\mathrm{d}V_M| \,,$$

whenever one of these integrals is well-defined.

Note that the innermost integral on the right-hand side is only defined almost everywhere.

2.C.2 The double-fibration trick

We now describe the double-fibration trick, which consists in applying the coarea formula twice, for different fibrations. Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \to \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. We equip Σ with the restriction of the product metric on $M_1 \times M_2$ and denote by $|\mathrm{d}V_{M_1}|$, $|\mathrm{d}V_{M_2}|$ and $|\mathrm{d}V_\Sigma|$ the Riemannian measures on the corresponding manifolds. Finally, let $\pi_1 : \Sigma \to M_1$ and $\pi_2 : \Sigma \to M_2$ be the projections from Σ to each factor. Assuming that $r \leqslant \min(n_1, n_2)$, we have $\dim(\Sigma) = n_1 + n_2 - r \geqslant \max(n_1, n_2)$. Thus we can apply the coarea formula both to π_1 and π_2.

Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function, then:

$$\begin{aligned}\int_{y_1\in M_1} \left(\int_{x\in\pi_1^{-1}(y_1)} \phi(x) \,|\mathrm{d}V_{y_1}| \right) |\mathrm{d}V_{M_1}| &= \int_{x\in\Sigma} \phi(x) \left|\det{}^{\perp}(d_x\pi_1)\right| |\mathrm{d}V_\Sigma| \\ &= \int_{y_2\in M_2} \left(\int_{x\in\pi_2^{-1}(y_2)} \phi(x) \frac{\left|\det{}^{\perp}(d_x\pi_1)\right|}{\left|\det{}^{\perp}(d_x\pi_2)\right|} |\mathrm{d}V_{y_2}| \right) |\mathrm{d}V_{M_2}| \,, \end{aligned} \tag{2.C.1}$$

whenever one of these integrals is well-defined. Note that if $\left|\det{}^{\perp}(d_x\pi_2)\right|$ vanishes then $\pi_2(x)$ is a critical value of π_2, and the set of such critical values has measure 0 in M_2.

We would like the integrand on the right-hand side to depend on F rather than on π_1 and π_2. Let $\partial_1 F$ and $\partial_2 F$ denote the partial differentials of F with respect to the first and second variable respectively. For any $x = (x_1, x_2) \in \Sigma$,

$$T_x\Sigma = \{(v_1, v_2) \in T_{x_1}M_1 \times T_{x_2}M_2 \mid \partial_1 F(x) \cdot v_1 + \partial_2 F(x) \cdot v_2 = 0\} \,.$$

Lemma 2.C.2. *Let $x \in \Sigma$, then $\left|\det^\perp(d_x\pi_2)\right| = 0$ if and only if $\left|\det^\perp(\partial_1 F(x))\right| = 0$. Moreover,*

$$\left|\det^\perp(d_x\pi_1)\right| \left|\det^\perp(\partial_1 F(x))\right| = \left|\det^\perp(d_x\pi_2)\right| \left|\det^\perp(\partial_2 F(x))\right|. \qquad (2.C.2)$$

Proof. First note that $d_x F = \partial_1 F(x) \circ d_x\pi_1 + \partial_2 F(x) \circ d_x\pi_2 = 0$ on $T_x\Sigma$. This shows that:

$$\ker(d_x\pi_1) = \{0\} \times \ker(\partial_2 F(x)) \qquad \text{and} \qquad \ker(d_x\pi_2) = \ker(\partial_1 F(x)) \times \{0\}. \qquad (2.C.3)$$

The space $T_x\Sigma$ splits as the following orthogonal direct sum:

$$T_x\Sigma = \ker(d_x\pi_1) \oplus \ker(d_x\pi_2) \oplus G, \qquad (2.C.4)$$

where G is the orthogonal complement of $\ker(d_x\pi_1) \oplus \ker(d_x\pi_2)$ in $T_x\Sigma$.

Then, $\left|\det^\perp(d_x\pi_2)\right| = 0$ if and only if $d_x\pi_2$ is not onto. Recalling that $\dim(M_2) = n_2$ and $\dim(\Sigma) = n_1 + n_2 - r$, this is equivalent to $\dim(\ker(d_x\pi_2)) > n_1 - r$. In the same way, $\left|\det^\perp(\partial_1 F(x))\right| = 0$ if and only if $\dim(\ker(\partial_1 F(x))) > n_1 - r$. But the kernels of $d_x\pi_2$ and $\partial_1 F(x)$ have the same dimension by (2.C.3), so that $\left|\det^\perp(d_x\pi_2)\right| = 0$ if and only if $\left|\det^\perp(\partial_1 F(x))\right| = 0$. A similar argument shows that $\left|\det^\perp(d_x\pi_1)\right| = 0$ if and only if $\left|\det^\perp(\partial_2 F(x))\right| = 0$. Thus the lemma is true if any of the four maps in (2.C.2) is singular.

From now on, we assume that these maps are all surjective. In this case, we have:

$$\dim(\ker(\partial_2 F(x))) = \dim(\ker(d_x\pi_1)) = n_2 - r,$$
$$\dim(\ker(\partial_1 F(x))) = \dim(\ker(d_x\pi_2)) = n_1 - r,$$

and

$$\dim(G) = r.$$

We choose an orthonormal basis of T_xM_1 adapted to $\ker(\partial_1 F(x)) \oplus \ker(\partial_1 F(x))^\perp$ and an orthonormal basis of T_xM_2 adapted to $\ker(\partial_2 F(x)) \oplus \ker(\partial_2 F(x))^\perp$. From these, we deduce orthonormal bases of

$$\ker(d_x\pi_1) = \{0\} \times \ker(\partial_2 F(x)) \qquad \text{and} \qquad \ker(d_x\pi_2) = \ker(\partial_1 F(x)) \times \{0\}.$$

Finally we complete the resulting basis of $\ker(d_x\pi_1) \oplus \ker(d_x\pi_2)$ in an orthonormal basis of $T_x\Sigma$ adapted to the splitting (2.C.4). In these bases the matrix of $d_x\pi_1$ has the form $\begin{pmatrix} 0 & I_{n_1-r} & A_1 \\ 0 & 0 & B_1 \end{pmatrix}$ where I_{n_1-r} stands for the identity matrix of size $n_1 - r$. Similarly, the matrix of $d_x\pi_2$ has the form $\begin{pmatrix} I_{n_2-r} & 0 & A_2 \\ 0 & 0 & B_2 \end{pmatrix}$, and the matrices of $\partial_1 F(x)$ and $\partial_2 F(x)$ have the form $\begin{pmatrix} 0 & C_1 \end{pmatrix}$ and $\begin{pmatrix} 0 & C_2 \end{pmatrix}$ respectively. Thus B_1, B_2, C_1 and C_2 are square matrices satisfying the following relations:

$$\left|\det^\perp(d_x\pi_1)\right| = \left|\det(B_1)\right|, \qquad \left|\det^\perp(\partial_1 F(x))\right| = \left|\det(C_1)\right|,$$
$$\left|\det^\perp(d_x\pi_2)\right| = \left|\det(B_2)\right|, \qquad \left|\det^\perp(\partial_2 F(x))\right| = \left|\det(C_2)\right|.$$

Besides the relation $\partial_1 F(x) \circ d_x\pi_1 + \partial_2 F(x) \circ d_x\pi_2 = 0$ means that $C_1 B_1 = -C_2 B_2$, hence $\left|\det(C_1)\right| \left|\det(B_1)\right| = \left|\det(C_2)\right| \left|\det(B_2)\right|$. This proves (2.C.2). □

An immediate consequence of (2.C.1) and (2.C.2) is the following.

Proposition 2.C.3. *Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \to \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{y_1 \in M_1} \left(\int_{\pi_1^{-1}(y_1)} \phi(x) \left|\mathrm{d}V_{y_1}\right| \right) \left|\mathrm{d}V_{M_1}\right| =$$
$$\int_{y_2 \in M_2} \left(\int_{\pi_2^{-1}(y_2)} \phi(x) \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} \left|\mathrm{d}V_{y_2}\right| \right) \left|\mathrm{d}V_{M_2}\right|,$$

whenever one of these integrals is well-defined.

2.C.3 Proof of Theorem 2.5.3

Finally, we prove Theorem 2.5.3. Let M be a closed Riemannian manifold of dimension n and V be a subspace of $\mathcal{C}^\infty(M, \mathbb{R}^r)$ of dimension N (recall that $1 \leqslant r \leqslant n$). We assume that V is 0-ample, so that $F : (f, x) \mapsto f(x)$ is a smooth submersion from $V \times M$ to $\mathbb{R}^r$ and

$$\Sigma = F^{-1}(0) = \{(f, x) \in V \times M \mid f(x) = 0\}$$

is a submanifold of codimension r of $V \times M$. Let $\mathrm{d}f$ denote the Lebesgue measure on V or on a subspace of V. Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function, by Proposition 2.C.3,

$$\begin{aligned}\mathbb{E}\left[\int_{x \in Z_f} \phi(f, x) \left|\mathrm{d}V_f\right|\right] &= \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{f \in V} \left(\int_{x \in Z_f} \phi(f, x) e^{-\frac{\|f\|^2}{2}} \left|\mathrm{d}V_f\right| \right) \mathrm{d}f \\ &= \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{x \in M} \left(\int_{f \in \ker(j_x^0)} \phi(f, x) e^{-\frac{\|f\|^2}{2}} \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} \mathrm{d}f \right) \left|\mathrm{d}V_M\right|.\end{aligned}$$

Recall that for all $x \in M$, $j_x^0 : f \mapsto f(x)$ is onto, since V is 0-ample. In particular, $\ker(j_x^0)$ has codimension r and V splits as $\ker(j_x^0) \oplus \ker(j_x^0)^\perp$. We recognize the innermost integral to be a conditional expectation given $f(x) = 0$ (see Corollary 2.A.11). Thus,

$$\mathbb{E}\left[\int_{x \in Z_f} \phi(f, x) \left|\mathrm{d}V_f\right|\right] = \frac{1}{(2\pi)^{\frac{r}{2}}} \int_{x \in M} \mathbb{E}\left[\phi(f, x) \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} \,\middle|\, f(x) = 0\right] \left|\mathrm{d}V_M\right|. \tag{2.C.5}$$

By equation (2.2.4), $\left|\det{}^{\perp}(\partial_2 F(x))\right| = \left|\det{}^{\perp}(d_x f)\right|$ and

$$\left|\det{}^{\perp}(\partial_1 F(x))\right| = \left|\det{}^{\perp}\left(j_x^0\right)\right| = \sqrt{\det\left(j_x^0 (j_x^0)^*\right)}. \tag{2.C.6}$$

By equation (2.A.8), since $f \sim \mathcal{N}(0, \mathrm{Id})$, $j_x^0 j_x^{0*}$ is the variance operator of $f(x) = j_x^0(f)$. Then, by (2.2.9), $\det\left(j_x^0 (j_x^0)^*\right) = \det(\mathrm{Var}(f(x))) = \det(E(x, x))$. In particular, this quantity does not depend on $f \in V$. Equations (2.C.5), (2.C.6) and this last equality prove Theorem 2.5.3.

CHAPITRE 3

Variance du volume des sous-variétés algébriques réelles aléatoires

Ce chapitre est consacré à l'étude de la variance du volume des sous-variétés algébriques réelles aléatoires décrites à la section 1.3. Plus généralement, on y étudie la variance des statistiques linéaires associées à ces sous-variétés. Il contient la preuve du théorème 1.3.19.

Ce chapitre est la reproduction de la prépublication [Let16b] :
T. Letendre, *Variance of the volume of random real algebraic submanifolds*, arXiv:1608.05658, (2016).

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3.1 Introduction

Framework. Let us first describe our framework and state the main results of this article (see Section 3.2 for more details). Let $\mathcal{X}$ be a smooth complex projective manifold of positive complex dimension n. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ and let $\mathcal{E}$ be a rank r holomorphic vector bundle over $\mathcal{X}$, with $r \in \{1, \dots, n\}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures and that the real locus M of $\mathcal{X}$ is not empty. Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on $\mathcal{E}$ and $\mathcal{L}$ respectively that are compatible with the real structures. We assume that $h_{\mathcal{L}}$ has positive curvature ω. Then ω is a Kähler form on $\mathcal{X}$ and it induces a Riemannian metric g on M.

For any $d \in \mathbb{N}$, the Kähler form ω, $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ induce a L^2-inner product on the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ (see (3.2.1)). Let $d \in \mathbb{N}$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by Z_s the real zero set $s^{-1}(0) \cap M$ of s. For d large enough, for almost every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, Z_s is a codimension r smooth submanifold of M and we denote by $|\mathrm{d}V_s|$ the Riemannian measure on Z_s induced by g (see Sect. 3.2.2). In the sequel, we will consider $|\mathrm{d}V_s|$ as a positive Radon measure on M. Let us also denote by $|\mathrm{d}V_M|$ the Riemannian measure on M.

Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then $|\mathrm{d}V_{s_d}|$ is a random positive Radon measure on M. We set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$ to avoid too many subscripts. In a previous paper [Let16a, thm. 2.1.3], we computed the asymptotic of the expected Riemannian volume of Z_d as $d \to +\infty$. Namely, we proved that:

$$\mathbb{E}[\mathrm{Vol}\,(Z_d)] = d^{\frac{r}{2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^{n})} + O\Big(d^{\frac{r}{2}-1}\Big)\,, \tag{3.1.1}$$

where $\mathrm{Vol}\,(M)$ is the volume of M for $|\mathrm{d}V_M|$ and the volumes of spheres are Euclidean volumes. Here and in throughout this paper, $\mathbb{E}[\cdot]$ denotes the expectation of the random variable between the brackets, and $\mathbb{S}^m$ stands for the unit Euclidean sphere of dimension m.

Let $\phi \in \mathcal{C}^0(M)$, we denote by $\|\phi\|_\infty = \max_{x\in M}|\phi(x)|$ its norm sup. Besides, we denote by $\langle\cdot\,,\cdot\rangle$ the duality pairing between $\big(\mathcal{C}^0(M), \|\cdot\|_\infty\big)$ and its topological dual. Then, (3.1.1) can be restated as:

$$\mathbb{E}[\langle|\mathrm{d}V_d|\,,\mathbf{1}\rangle] = d^{\frac{r}{2}}\,\mathrm{Vol}\,(M)\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^{n})} + O\Big(d^{\frac{r}{2}-1}\Big)\,,$$

where $\mathbf{1} \in \mathcal{C}^0(M)$ stands for the unit constant function on M. The same proof gives similar asymptotics for $\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi\rangle]$ for any continuous $\phi : M \to \mathbb{R}$ (see [Let16a, section 2.5.3]).

Theorem 3.1.1. *Let $\mathcal{X}$ be a complex projective manifold of positive dimension n defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank r Hermitian vector bundle with $1 \leqslant r \leqslant n-1$ and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then the following holds as $d \to +\infty$:*

$$\forall \phi \in \mathcal{C}^0(M), \qquad \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi\rangle] = d^{\frac{r}{2}}\left(\int_M \phi\,|\mathrm{d}V_M|\right)\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^{n})} + \|\phi\|_\infty\, O\Big(d^{\frac{r}{2}-1}\Big)\,. \tag{3.1.2}$$

Moreover the error term $O\Big(d^{\frac{r}{2}-1}\Big)$ does not depend on ϕ.

In particular, we can define a sequence of Radon measures $(\mathbb{E}[|\mathrm{d}V_d|])_{d\geqslant d_0}$ on M by: for every $d \geqslant d_0$ and every $\phi \in \mathcal{C}^0(M)$, $\langle\mathbb{E}[|\mathrm{d}V_d|]\,,\phi\rangle = \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi\rangle]$. Then Thm. 3.1.1 implies that:

$$\Big(d^{-\frac{r}{2}}\Big)\,\mathbb{E}[|\mathrm{d}V_d|] \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^{n})}\,|\mathrm{d}V_M|\,, \tag{3.1.3}$$

as continuous linear functionals on $\big(\mathcal{C}^0(M), \|\cdot\|_\infty\big)$.

Statement of the results. The main result of this paper is an asymptotic for the covariances of the linear statistics $\big\{\langle|\mathrm{d}V_d|\,,\phi\rangle \bigm| \phi \in \mathcal{C}^0(M)\big\}$. Before we can state our theorem, we need to introduce some additional notations.

As usual, we denote by $\mathrm{Var}(X) = \mathbb{E}\Big[(X-\mathbb{E}[X])^2\Big]$ the variance of the real random variable X, and by $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])\,(Y-\mathbb{E}[Y])]$ the covariance of the real random variables X and Y. We call *variance* of $|\mathrm{d}V_d|$ and we denote by $\mathrm{Var}(|\mathrm{d}V_d|)$ the symmetric bilinear form on $\mathcal{C}^0(M)$ defined by:

$$\forall \phi_1,\phi_2 \in \mathcal{C}^0(M), \qquad \mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2) = \mathrm{Cov}(\langle|\mathrm{d}V_d|\,,\phi_1\rangle\,,\langle|\mathrm{d}V_d|\,,\phi_2\rangle)\,. \tag{3.1.4}$$

Definition 3.1.2. Let $\phi \in \mathcal{C}^0(M)$, we denote by ϖ_ϕ its *continuity modulus*, which is defined by:

$$\begin{array}{rccc} \varpi_\phi : & (0,+\infty) & \longrightarrow & [0,+\infty) \\ & \varepsilon & \longmapsto & \sup\big\{|\phi(x)-\phi(y)| \bigm| (x,y)\in M^2, \rho_g(x,y)\leqslant\varepsilon\big\}\,, \end{array}$$

where $\rho_g(\cdot,\cdot)$ stands for the geodesic distance on (M,g).

Since M is compact, ϖ_ϕ is well-defined for every $\phi \in \mathcal{C}^0(M)$. Moreover every $\phi \in \mathcal{C}^0(M)$ is uniformly continuous and we have:

$$\forall \phi \in \mathcal{C}^0(M), \qquad \varpi_\phi(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

Note that, if $\phi : M \to \mathbb{R}$ is Lipschitz continuous, then $\varpi_\phi(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.

Definition 3.1.3. Let $L : V \to V'$ be a linear map between two Euclidean spaces, we denote by $\left|\det^\perp(L)\right|$ the *Jacobian* of L:

$$\left|\det^\perp(L)\right| = \sqrt{\det(LL^*)},$$

where $L^* : V' \to V$ is the adjoint operator of L.

See Section 3.4.1 for a quick discussion of the properties of this Jacobian. If A is an element of $\mathcal{M}_{rn}(\mathbb{R})$, the space of matrices of size $r \times n$ with real coefficients, we denote by $\left|\det^\perp(A)\right|$ the Jacobian of the linear map from $\mathbb{R}^n$ to $\mathbb{R}^r$ associated to A in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^r$.

Definition 3.1.4. For every $t > 0$, we define $(X(t), Y(t))$ to be a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ with variance matrix:

$$\left(\begin{array}{ccccc|ccccc}
1-\frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}}-\frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & \vdots & 0 & e^{-\frac{t}{2}} & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & 0 & \vdots & & \ddots & e^{-\frac{t}{2}} & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} \\
\hline
e^{-\frac{t}{2}}-\frac{te^{-\frac{t}{2}}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 & 1-\frac{te^{-t}}{1-e^{-t}} & 0 & \cdots & \cdots & 0 \\
0 & e^{-\frac{t}{2}} & \ddots & & \vdots & 0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & e^{-\frac{t}{2}} & 0 & \vdots & & \ddots & 1 & 0 \\
0 & \cdots & \cdots & 0 & e^{-\frac{t}{2}} & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) \otimes I_r,$$

where I_r is the identity matrix of size r. That is, if we denote by $X_{ij}(t)$ (resp. $Y_{ij}(t)$) the coefficients of $X(t)$ (resp. $Y(t)$), the couples $(X_{ij}(t), Y_{ij}(t))$ with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$ are independent from one another and the variance matrix of $(X_{ij}(t), Y_{ij}(t))$ is:

$$\begin{pmatrix} 1-\frac{te^{-t}}{1-e^{-t}} & e^{-\frac{t}{2}}\left(1-\frac{t}{1-e^{-t}}\right) \\ e^{-\frac{t}{2}}\left(1-\frac{t}{1-e^{-t}}\right) & 1-\frac{te^{-t}}{1-e^{-t}} \end{pmatrix} \quad \text{if } j=1, \text{ and} \quad \begin{pmatrix} 1 & e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} & 1 \end{pmatrix} \quad \text{otherwise.}$$

Notation 3.1.5. We set $\alpha_0 = \dfrac{n-r}{2(2r+1)(2n+1)}$.

We can now state our main result.

Theorem 3.1.6. *Let $\mathcal{X}$ be a complex projective manifold of dimension $n \geqslant 2$ defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank r Hermitian vector bundle with $1 \leqslant r \leqslant n-1$ and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.*

Let $\beta \in (0, \frac{1}{2})$, then there exists $C_\beta > 0$ such that, for all $\alpha \in (0, \alpha_0)$, for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:

$$\begin{aligned}\operatorname{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = d^{r-\frac{n}{2}} \left(\int_M \phi_1\phi_2 \,|\mathrm{d}V_M| \right) \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} \\ + \|\phi_1\|_\infty \|\phi_2\|_\infty O\Big(d^{r-\frac{n}{2}-\alpha}\Big) + \|\phi_1\|_\infty \varpi_{\phi_2} \left(C_\beta d^{-\beta}\right) O\Big(d^{r-\frac{n}{2}}\Big), \quad (3.1.5)\end{aligned}$$

where

$$\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} \left(\frac{\mathbb{E}\left[\left|\det^\perp(X(t))\right| \left|\det^\perp(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r \left(\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r}\right)}{\operatorname{Vol}\left(\mathbb{S}^{n}\right)}\right)^2 \right) t^{\frac{n-2}{2}} \,\mathrm{d}t < +\infty. \quad (3.1.6)$$

Moreover the error terms $O\Big(d^{r-\frac{n}{2}-\alpha}\Big)$ and $O\Big(d^{r-\frac{n}{2}}\Big)$ in (3.1.5) do not depend on (ϕ_1, ϕ_2).

We obtain the variance of the volume of Z_d by applying Thm. 3.1.6 to $\phi_1 = \phi_2 = \mathbf{1}$. When $\phi_1 = \phi_2 = \phi$ we get the following.

Corollary 3.1.7 (Variance of the linear statistics)**.** *In the same setting as Thm. 3.1.6, let $\beta \in (0, \frac{1}{2})$, then there exists $C_\beta > 0$ such that, for all $\alpha \in (0, \alpha_0)$ and all $\phi \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:*

$$\begin{aligned}\operatorname{Var}(\langle|\mathrm{d}V_d|\,, \phi\rangle) = d^{r-\frac{n}{2}} \left(\int_M \phi^2 \,|\mathrm{d}V_M| \right) \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} \\ + \|\phi\|_\infty^2 O\Big(d^{r-\frac{n}{2}-\alpha}\Big) + \|\phi\|_\infty \varpi_{\phi} \left(C_\beta d^{-\beta}\right) O\Big(d^{r-\frac{n}{2}}\Big). \quad (3.1.7)\end{aligned}$$

Moreover, the error terms do not depend on ϕ.

Remarks 3.1.8. Some remarks are in order.

- The value of the constant α_0 should not be taken too seriously. This constant appears for technical reasons and it is probably far from optimal.

- If ϕ_2 is Lipschitz continuous with Lipschitz constant K, then the error term in eq. (3.1.5) can be replaced by:

  $$\|\phi_1\|_\infty (\|\phi_2\|_\infty + K)\, O\Big(d^{r-\frac{n}{2}-\alpha}\Big)$$

  by fixing $\beta > \alpha_0$, which is possible since $\frac{1}{2} > \alpha_0$.

- Thm. 3.1.6 shows that $\operatorname{Var}(|\mathrm{d}V_d|)$ is a continuous bilinear form on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ for d large enough. Moreover, denoting by $\langle \cdot\,, \cdot\rangle_M$ the L^2-inner product on $\mathcal{C}^0(M)$ defined by $\langle \phi_1\,, \phi_2\rangle_M = \displaystyle\int_M \phi_1\phi_2 \,|\mathrm{d}V_M|$, we have:

  $$d^{\frac{n}{2}-r} \operatorname{Var}(|\mathrm{d}V_d|) \xrightarrow[d\to+\infty]{} \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} \langle \cdot\,, \cdot\rangle_M$$

  in the weak sense. A priori, there is no such convergence as continuous bilinear forms on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ since the estimate (3.1.5) involves the continuity modulus of ϕ_2.

- The fact that the constant $\mathcal{I}_{n,r}$ is finite is part of the statement and is proved below (Lemma 3.4.25). This constant is necessarily non-negative. Numerical evidence suggests that it is positive but we do not know how to prove it at this point.

- Thm. 3.1.6 does not apply in the case of maximal codimension ($r = n$). This case presents an additional singularity which causes our proof to fail. However, we believe a similar result to be true for $r = n$, at least in the case of the Kostlan–Shub–Smale polynomials described below (compare [Dal15, Wsc05]).

Corollary 3.1.9 (Concentration in probability)**.** *In the same setting as Thm. 3.1.6, let $\alpha \geqslant \frac{r}{2} - \frac{n}{4}$ and let $\phi \in \mathcal{C}^0(M)$. Then, for every $\varepsilon > 0$, we have:*

$$\mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \,, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi\rangle]\right| > d^\alpha \varepsilon\right) = \frac{1}{\varepsilon^2} O\left(d^{r-\frac{n}{2}-2\alpha}\right),$$

where the error term is independent of ε, but depends on ϕ.

Corollary 3.1.10. *In the same setting as Thm. 3.1.6, let $U \subset M$ be an open subset, then as $d \to +\infty$ we have:*

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Our last corollary is concerned with the convergence of a random sequence of sections of increasing degree. Let us denote by $\mathrm{d}\nu_d$ the standard Gaussian measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (see (3.2.4)). Let $\mathrm{d}\nu$ denote the product measure $\bigotimes_{d\in\mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then we have the following.

Corollary 3.1.11 (Almost sure convergence)**.** *In the same setting as Thm. 3.1.6, let us assume that $n \geqslant 3$. Let $(s_d)_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a random sequence of sections. Then, $\mathrm{d}\nu$-almost surely, we have:*

$$\forall \phi \in \mathcal{C}^0(M), \qquad d^{-\frac{r}{2}} \langle |\mathrm{d}V_{s_d}| \,, \phi\rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left(\int_M \phi \,|\mathrm{d}V_M|\right).$$

That is, $\mathrm{d}\nu$-almost surely,

$$d^{-\frac{r}{2}} |\mathrm{d}V_{s_d}| \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} |\mathrm{d}V_M|,$$

in the sense of the weak convergence of measures.

Remark 3.1.12. We expect this result to hold for $n = 2$ as well, but our proof fails in this case.

The Kostlan–Shub–Smale polynomials Let us consider the simplest example of our framework. We choose $\mathcal{X}$ to be the complex projective space $\mathbb{CP}^n$, with the real structure defined by the usual conjugation in $\mathbb{C}^{n+1}$. Then M is the real projective space $\mathbb{RP}^n$. Let $\mathcal{L} = \mathcal{O}(1)$ be the hyperplane line bundle, equipped with its natural real structure and the metric dual to the standard metric on the tautological line bundle over $\mathbb{CP}^n$. Then the curvature form of $\mathcal{L}$ is the Fubini–Study form ω_{FS}, normalized so that the induced Riemannian metric is the quotient of the Euclidean metric on the unit sphere of $\mathbb{C}^{n+1}$. Let $\mathcal{E} = \mathbb{C}^r \times \mathbb{CP}^n \to \mathbb{CP}^n$ be the rank r trivial bundle with the trivial real structure and the trivial metric.

In this setting, the global holomorphic sections of $\mathcal{L}^d$ are the complex homogeneous polynomials of degree d in $n+1$ variables and those of $\mathcal{E} \otimes \mathcal{L}^d$ are r-tuples of such polynomials, since $\mathcal{E}$ is trivial. Finally, the real structures being just the usual conjugations, we have:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]^r,$$

where $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ is the space of real homogeneous polynomials of degree d in $n+1$ variables. The r copies of this space in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ are pairwise orthogonal for the inner product (3.2.1). Hence a standard Gaussian in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is a r-tuple of independent standard Gaussian in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n] = \mathbb{R}H^0\left(\mathcal{X}, \mathcal{L}^d\right)$.

It is well-known (cf. [BSZ00, BBL96, Kos93]) that the monomials are pairwise orthogonal for the L^2-inner product (3.2.1), but not orthonormal. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we denote its length by $|\alpha| = \alpha_0 + \cdots + \alpha_n$. We also define $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ and $\alpha! = (\alpha_0!) \cdots (\alpha_n!)$. Finally, if $|\alpha| = d$, we denote by $\binom{d}{\alpha}$ the multinomial coefficient $\frac{d!}{\alpha!}$. Then, an orthonormal basis of $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ for the inner product (3.2.1) is given by the family:

$$\left(\sqrt{\frac{(d+n)!}{\pi^n d!}} \sqrt{\binom{d}{\alpha}} X^\alpha\right)_{|\alpha|=d}.$$

Thus a standard Gaussian vector in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ is a random polynomial:

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where the coefficients $(a_\alpha)_{|\alpha|=d}$ are independent real standard Gaussian variables. Since we are only concerned with the zero set of this random polynomial, we can drop the factor $\sqrt{\frac{(d+n)!}{\pi^n d!}}$.

Finally, in this setting, $|\mathrm{d}V_d|$ is the common zero set of r independent random polynomials in $\mathbb{R}^d_{\mathrm{hom}}[X_0, \dots, X_n]$ of the form:

$$\sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha, \tag{3.1.8}$$

with independent coefficients $(a_\alpha)_{|\alpha|=d}$ distributed according to the real standard Gaussian distribution. Such polynomials are known as the Kostlan–Shub–Smale polynomials. They were introduced in [Kos93, SS93] and were actively studied since (cf. [AW05, Bü07, Dal15, Pod01, Wsc05]).

Related works. As we just said, zero sets of systems of independent random polynomials distributed as (3.1.8) were studied by Kostlan [Kos93] and Shub and Smale [SS93]. The expected volume of these random algebraic manifolds was computed by Kostlan [Kos93] and their expected Euler characteristic was computed by Podkorytov [Pod01] in codimension 1, and by Bürgisser [Bü07] in higher codimension. Both these results were extended to the setting of the present paper in [Let16a].

In [Wsc05], Wschebor obtained an asymptotic bound, as the dimension n goes to infinity, for the variance of number of real roots of a system of n independent Kostlan–Shub–Smale polynomials. Recently, Dalmao [Dal15] computed an asymptotic of order $\sqrt{d}$ for the variance of the number of real roots of one Kostlan–Shub–Smale polynomial in dimension $n = 1$. His result is very similar to (3.1.5), which leads us to think that such a result should hold for $r = n$. He also proved a central limit theorem for this number of real roots, using Wiener chaos methods.

In [KL01, thm. 3], Kratz and Leòn considered the level curves of a centered stationary Gaussian field with unit variance on the plane $\mathbb{R}^2$. More precisely, they considered the length of a level curve intersected with some large square $[-T, T] \times [-T, T]$. As $T \to +\infty$, they proved asymptotics of order T^2 for both the expectation and the variance of this length. They also proved that it satisfies a central limit theorem as $T \to +\infty$. In particular, their result applies to the centered Gaussian field on $\mathbb{R}^2$ with correlation function:

$$(x, y) \longmapsto \exp\left(-\frac{1}{2} \|x - y\|^2\right).$$

This field can be seen as the scaling limit, in the sense of [NS15], of the centered Gaussian field $(s_d(x))_{x\in M}$ defined by our random sections, when $n = 2$ and $r = 1$.

The study of more general random algebraic submanifolds, obtained as the zero sets of random sections, was pioneered by Shiffman and Zelditch [SZ99, SZ08, SZ10]. They considered the integration current over the common complex zero set Z_d of r independent random sections in $H^0(\mathcal{X}, \mathcal{L}^d)$, distributed as standard complex Gaussians. In [SZ99], they computed the asymptotic, as d goes to infinity, of the expectation of the associated smooth statistics when $r = 1$. They also provided an upper bound for the variance of these quantities and proved the equivalent of Cor. 3.1.11 in this complex algebraic setting. In [SZ08], they gave an asymptotic of order $d^{2r-n-\frac{1}{2}}$ for the variance of the volume of $Z_d \cap U$, where $U \subset \mathcal{X}$ is a domain satisfying some regularity conditions. In [SZ10], they proved a similar asymptotic for the variance of the smooth statistics associated to Z_d. When $r = 1$, they deduced a central limit theorem from these estimates and an asymptotic normality result of Sodin and Tsirelson [ST04]. Finally, in [SZZ08, thm. 1.4], Shiffman, Zelditch and Zrebiec proved that the probability that $Z_d \cap U = \emptyset$, where U is any open subset of $\mathcal{X}$, decreases exponentially fast as d goes to infinity.

Coming back to our real algebraic setting, one should be able to deduce from the general result of Nazarov and Sodin [NS15, thm. 3] that, given an open set $U \subset M$, the probability that $Z_d \cap U = \emptyset$ goes to 0 as d goes to infinity. Corollary 3.1.10 gives an upper bound for the convergence rate. In particular, this bounds the probability for Z_d to be empty. In the same spirit, Gayet and Welschinger [GW15a] proved the following result. Let Σ be a fixed diffeomorphism type of codimension r submanifold of $\mathbb{R}^n$, let $x \in M$ and let $B_d(x)$ denote the geodesic ball of center x and radius $\frac{1}{\sqrt{d}}$. Then, the probability that $Z_d \cap B_d(x)$ contains a submanifold diffeomorphic to Σ is bounded from below. On the other hand, when $n = 2$ and $r = 1$, the Harnack–Klein inequality shows that the number of connected components of Z_d is bounded by a polynomial in d. In [GW11a], Gayet and Welschinger proved that the probability for Z_d to have the maximal number of connected components decreases exponentially fast with d.

Another well-studied model of random submanifolds is that of Riemannian random waves, i.e. zero sets of random eigenfunctions of the Laplacian associated to some eigenvalue λ. In this setting, Rudnick and Wigman [RW08] computed an asymptotic bound, as $\lambda \to +\infty$, for the variance of the volume of a random hypersurface on the flat n-dimensional torus $\mathbb{T}^n$. On $\mathbb{T}^2$, this result was improved by Krishnapur, Kurlberg and Wigman [KKW13] who computed the precise asymptotic of the variance of the lenght of a random curve. In [Wig10], Wigman computed the asymptotic variance of the linear statistics associated to a random curve on the Euclidean sphere $\mathbb{S}^2$. His result holds for a large class of test-function that contains the characteristic functions of open sets satisfying some regularity assumption.

In relation with Cor. 3.1.10, Nazarov and Sodin [NS09] proved that, on the Euclidean sphere $\mathbb{S}^2$, the number of connected components of a random curve times $\frac{1}{\lambda}$ converges exponentially fast in probability to a deterministic constant as $\lambda \to +\infty$.

About the proof. The idea of the proof is the following. The random section s_d defines a centered Gaussian field $(s_d(x))_{x\in\mathcal{X}}$. The correlation kernel of this field equals the Bergman kernel, that is the kernel of the orthogonal projection onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ for the inner product (3.2.1) (compare [BSZ00, Let16a, SZ99, SZ08, SZ10]).

In order to compute the covariance of the smooth statistics $\langle |\mathrm{d}V_s| , \phi_1\rangle$ and $\langle |\mathrm{d}V_s| , \phi_2\rangle$, we apply a Kac–Rice formula (cf. [AW09, BSZ00, Dal15, TA07, Wig10]). This allows us to write $\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2)$ as the integral over $M \times M$ of some function $\mathcal{D}_d(x, y)$, defined by (3.4.9). This density $\mathcal{D}_d(x,y)$ is the difference of two terms, coming respectively from

$$\mathbb{E}[\langle |\mathrm{d}V_d| \ ,\phi_1\rangle \langle |\mathrm{d}V_d| \ ,\phi_2\rangle] \qquad \text{and} \qquad \mathbb{E}[\langle |\mathrm{d}V_d| \ ,\phi_1\rangle]\, \mathbb{E}[\langle |\mathrm{d}V_d| \ ,\phi_2\rangle]\,.$$

Since the Bergman kernel decreases exponentially fast outside of the diagonal Δ in M^2 (see Section 3.3.4), the values of $s_d(x)$ and $s_d(y)$ are almost uncorrelated for (x,y) far from Δ. As a consequence, when the distance between x and y is much larger than $\frac{1}{\sqrt{d}}$, the above two terms in the expression of $\mathcal{D}_d(x,y)$ are equal, up to a small error (see Sect. 3.4.3 for a precise statement). Thus, $\mathcal{D}_d(x,y)$ is small far from Δ, and its integral over this domain only contributes a remainder term to $\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2)$.

The main contribution to the value of $\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2)$ comes from the integration of $\mathcal{D}_d(x,y)$ over a neighborhood of Δ of size about $\frac{1}{\sqrt{d}}$. We perform a change of variable in order to express this term as an integral over a domain of fixed size. This rescaling by $\frac{1}{\sqrt{d}}$ explains the factor $d^{-\frac{n}{2}}$ in (3.1.5). Besides, the order of growth of $\mathcal{D}_d(x,y)$ close to Δ is d^r, that is the order of growth of the square of $\mathbb{E}[|\mathrm{d}V_d|]$ (see Thm. 3.1.1). Finally, we get an order of growth of $d^{r-\frac{n}{2}}$ for $\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2)$. The constant in (3.1.5) appears as the scaling limit of the integral of $\mathcal{D}_d(x,y)$ over a neighborhood of Δ of typical size $\frac{1}{\sqrt{d}}$.

The difficulty in making this sketch of proof rigorous comes from the combination of two facts. First, we do not know exactly the value of the Bergman kernel (our correlation function) and its derivatives, but only asymptotics. In addition, the conditioning in the Kac–Rice formula is singular along Δ, and so is $\mathcal{D}_d$. Because of this, we lose all uniformity in the control of the error terms close to the diagonal. Nonetheless, by careful bookkeeping of the error terms, we can make the above heuristic precise.

Outline of the paper. In Section 3.2 we describe precisely our framework and the construction of the random measures $|\mathrm{d}V_{s_d}|$. We also introduce the Bergman kernel and explain how it is related to our random submanifolds.

In Section 3.3, we recall various estimates for the Bergman kernel that we use in the proof of our main theorem. These estimates were established by Dai, Liu and Ma [DLM06], and Ma and Marinescu [MM07, MM13, MM15] in a complex setting. Our main contribution in this section consists in checking that the preferred trivialization used by Ma and Marinescu to state their near-diagonal estimates is well-behaved with respect to the real structures on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ (see Section 3.3.1).

Section 3.4 is concerned with the proof of Thm. 3.1.6. In Sect. 3.4.1, we prove a Kac–Rice formula adapted to our problem, using Federer's coarea formula and Kodaira's embedding theorem. In Sect. 3.4.2 we prove an integral formula for the variance, using the Kac–Rice formula (Thm. 3.4.4). The core of the proof is contained in Sect. 3.4.3.

Finally, we prove Corollaries 3.1.9, 3.1.10 and 3.1.11 in Section 3.5.

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3.2 Random real algebraic submanifolds

3.2.1 General setting

In this section, we introduce our framework. It is the same as the algebraic setting of [Let16a], see also [GW15a, GW16]. Classical references for the material of this section are [GH94, chap. 0] and [Sil89, chap. 1].

Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension $n \geqslant 2$. We assume that $\mathcal{X}$ is defined over the reals, that is $\mathcal{X}$ is equipped with an anti-holomorphic involution $c_{\mathcal{X}}$. The real locus of $(\mathcal{X}, c_{\mathcal{X}})$ is the set of fixed points of $c_{\mathcal{X}}$. In the sequel, we assume that it is non-empty and we denote it by M. It is a classical fact that M is a smooth closed (i.e. compact without boundary) submanifold of $\mathcal{X}$ of real dimension n (see [Sil89, chap. 1]).

Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle of rank $r \in \{1, \dots, n-1\}$. Let $c_{\mathcal{E}}$ be a real structure on $\mathcal{E}$, compatible with $c_{\mathcal{X}}$ in the sense that the projection $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{X}$ satisfies $c_{\mathcal{X}} \circ \pi_{\mathcal{E}} = \pi_{\mathcal{E}} \circ c_{\mathcal{E}}$ and $c_{\mathcal{E}}$ is fiberwise $\mathbb{C}$-anti-linear. Let $h_{\mathcal{E}}$ be a real Hermitian metric on $\mathcal{E}$, that is $c_{\mathcal{E}}^{\star}(h_{\mathcal{E}}) = \overline{h_{\mathcal{E}}}$.

Similarly, let $\mathcal{L} \to \mathcal{X}$ be an ample holomorphic line bundle equipped with a compatible real structure $c_{\mathcal{L}}$ and a real Hermitian metric $h_{\mathcal{L}}$. Moreover, we assume that the curvature form ω of $h_{\mathcal{L}}$ is a Kähler form. Recall that if ζ is any non-vanishing holomorphic section on the open set $\Omega \subset \mathcal{X}$, then the restriction of ω to Ω is given by:

$$\omega_{/\Omega} = \frac{1}{2i}\partial\bar{\partial} \ln\left(h_{\mathcal{L}}(\zeta, \zeta)\right).$$

This Kähler form is associated to a Hermitian metric $g_{\mathbb{C}}$ on $\mathcal{X}$. The real part of $g_{\mathbb{C}}$ defines a Riemannian metric $g = \omega(\cdot, i\cdot)$ on $\mathcal{X}$, compatible with the complex structure. Note that, since $h_{\mathcal{L}}$ is compatible with the real structures on $\mathcal{X}$ and $\mathcal{L}$, we have $c_{\mathcal{L}}^{\star}(h_{\mathcal{L}}) = \overline{h_{\mathcal{L}}}$ and $c_{\mathcal{X}}^{\star}\omega = -\omega$. Then we have $c_{\mathcal{X}}^{\star} g_{\mathbb{C}} = \overline{g_{\mathbb{C}}}$, hence $c_{\mathcal{X}}^{\star} g = g$ and $c_{\mathcal{X}}$ is an isometry of $(\mathcal{X}, g)$.

Then g induces a Riemannian measure on every smooth submanifold of $\mathcal{X}$. In the case of $\mathcal{X}$, this measure is given by the volume form $\mathrm{d}V_{\mathcal{X}} = \frac{\omega^n}{n!}$. We denote by $|\mathrm{d}V_M|$ the Riemannian measure on (M, g).

Let $d \in \mathbb{N}$, then the rank r holomorphic vector bundle $\mathcal{E} \otimes \mathcal{L}^d$ can be endowed with a real structure $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d$, compatible with $c_{\mathcal{X}}$, and a real Hermitian metric $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$. If $x \in M$, then c_d induces a $\mathbb{C}$-anti-linear involution of the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the fixed points set of this involution, which is a dimension r real vector space.

Let $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. We can define a Hermitian inner product on $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ by:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \qquad \langle s_1\,, s_2\rangle = \int_{\mathcal{X}} h_d(s_1(x), s_2(x))\,\mathrm{d}V_{\mathcal{X}}. \tag{3.2.1}$$

We say that a section $s \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is real if it is equivariant for the real structures, that is: $c_d \circ s = s \circ c_{\mathcal{X}}$. Let $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the real vector space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. The restriction of $\langle \cdot\,, \cdot \rangle$ to $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product.

Notation 3.2.1. In this paper, $\langle \cdot\,, \cdot \rangle$ will always denote either the inner product on the concerned Euclidean (or Hermitian) space or the duality pairing between a space and its topological dual. Which one will be clear from the context.

Let $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This space has finite complex dimension N_d by Hodge's theory (compare [MM07, thm. 1.4.1]). We denote by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ the space of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \;\middle|\; c_d \circ s = s \circ c_{\mathcal{X}} \right\}. \tag{3.2.2}$$

The restriction of the inner product (3.2.1) to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ makes it into a Euclidean space of real dimension N_d.

Remark 3.2.2. Note that, even when we consider real sections restricted to M, the inner product is defined by integrating on the whole complex manifold $\mathcal{X}$.

3.2.2 Random submanifolds

This section is concerned with the definition of the random submanifolds we consider and the related random variables.

Let $d \in \mathbb{N}$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, we denote the real zero set of s by $Z_s = s^{-1}(0)\cap M$. If the restriction of s to M vanishes transversally, then Z_s is a smooth submanifold of codimension r of M. In this case, we denote by $|\mathrm{d}V_s|$ the Riemannian measure on Z_s induced by g, seen as a Radon measure on M. Note that this includes the case where Z_s is empty.

Recall the following facts, that we already discussed in [Let16a].

Definition 3.2.3 (see [Nic15c]). We say that $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ is 0-*ample* if, for any $x \in M$, the evaluation map

$$\begin{array}{rccc} \mathrm{ev}_x^d: & \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) & \longrightarrow & \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \\ & s & \longmapsto & s(x) \end{array} \tag{3.2.3}$$

is surjective.

Lemma 3.2.4 (see [Let16a], cor. 2.3.10). *There exists $d_1 \in \mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$, such that for all $d \geqslant d_1$, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ is* 0*-ample.*

Lemma 3.2.5 (see [Let16a], section 2.2.6). *If $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ is* 0*-ample, then for almost every section $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ (for the Lebesgue measure), the restriction of s to M vanishes transversally.*

From now on, we only consider the case $d \geqslant d_1$, so that $|\mathrm{d}V_s|$ is a well-defined measure on M for almost every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, that is s_d is a random vector which distribution admits the density:

$$s \mapsto \frac{1}{\sqrt{2\pi}^{N_d}} \exp\left(-\frac{1}{2}\left\|s\right\|^2\right) \tag{3.2.4}$$

with respect to the normalized Lebesgue measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Here $\|\cdot\|$ is the norm associated to the Euclidean inner product (3.2.1). Then Z_{s_d} is almost surely a submanifold of codimension r of M and $|\mathrm{d}V_{s_d}|$ is almost surely a random positive Radon measure on M. To simplify notations, we set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$. For more details concerning Gaussian vectors, we refer to [Let16a, appendix 2.A] and the references therein.

Let $\phi \in \mathcal{C}^0(M)$, for every $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ vanishing transversally, we set

$$\langle |\mathrm{d}V_s| \,, \phi\rangle = \int_{x\in Z_s} \phi(x)\,|\mathrm{d}V_s|\,. \tag{3.2.5}$$

Such a ϕ will be refered to as a *test-function*. Following [SZ10], we call *linear statistic* of degree d associated to ϕ the real random variable $\langle |\mathrm{d}V_d| \,, \phi\rangle$.

3.2.3 The correlation kernel

Let $d \in \mathbb{N}$, then $(s_d(x))_{x\in\mathcal{X}}$ is a smooth centered Gaussian field on $\mathcal{X}$. As such, it is characterized by its correlation kernel. In this section, we recall that the correlation kernel of s_d equals the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$. This is now a well-known fact (see [BSZ00, GW16, SZ99, SZ10]) and was already used by the author in [Let16a].

Let us first recall some facts about random vectors (see for example [Let16a, app. 2.A]). In this paper, we only consider centered random vectors (that is their expectation vanishes), so we give the following definitions in this restricted setting. Let X_1 and X_2 be centered random vectors taking values in Euclidean (or Hermitian) vector spaces V_1 and V_2 respectively, then we define their *covariance operator* as:

$$\operatorname{Cov}(X_1, X_2) : v \longmapsto \mathbb{E}[X_1 \langle v , X_2\rangle] \tag{3.2.6}$$

from V_2 to V_1. For every $v \in V_2$, we set $v^* = \langle \cdot , v\rangle \in V_2^*$. Then $\operatorname{Cov}(X_1, X_2) = \mathbb{E}[X_1 \otimes X_2^*]$ is an element of $V_1 \otimes V_2^*$. The *variance operator* of a centered random vector $X \in V$ is defined as $\operatorname{Var}(X) = \operatorname{Cov}(X, X) = \mathbb{E}[X \otimes X^*] \in V \otimes V^*$. We denote by $X \sim \mathcal{N}(\Lambda)$ the fact that X is a centered Gaussian vector with variance operator Λ. Finally, we say that $X \in V$ is a *standard* Gaussian vector if $X \sim \mathcal{N}(\mathrm{Id})$, where Id is the identity operator on V. A standard Gaussian vector admits the density (3.2.4) with respect to the normalized Lebesgue measure on V.

Recall that $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$ stands for the bundle $P_1^\star\left(\mathcal{E}\otimes\mathcal{L}^d\right)\otimes P_2^\star\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*\right)$ over $\mathcal{X}\times\mathcal{X}$, where P_1 (resp. P_2) denotes the projection from $\mathcal{X}\times\mathcal{X}$ onto the first (resp. second) factor. The covariance kernel of $(s_d(x))_{x\in\mathcal{X}}$ is the section of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$ defined by:

$$(x, y) \mapsto \operatorname{Cov}(s_d(x), s_d(y)) = \mathbb{E}[s_d(x) \otimes s_d(y)^*]\,. \tag{3.2.7}$$

The orthogonal projection from $\mathbb{R}\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$ onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ admits a Schwartz kernel (cf. [MM07, thm. B.2.7]). That is, there exists a unique section of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$, denoted by E_d, such that for any $s \in \mathbb{R}\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$ the projection of s onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ is given by:

$$x \longmapsto \int_{y\in\mathcal{X}} E_d(x, y)\,(s(y))\,\mathrm{d}V_{\mathcal{X}}. \tag{3.2.8}$$

This section is called the *Bergman kernel* of $\mathcal{E}\otimes\mathcal{L}^d$. Note that E_d is also the Schwartz kernel of the orthogonal projection from $\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$ onto $H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, for the Hermitian inner product (3.2.1).

Proposition 3.2.6. *Let $d \in \mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then, for all x and $y \in \mathcal{X}$, we have:*

$$\operatorname{Cov}(s_d(x), s_d(y)) = \mathbb{E}[s_d(x) \otimes s_d(y)^*] = E_d(x, y). \tag{3.2.9}$$

Proof. We will prove that $(x, y) \mapsto \mathbb{E}[s_d(x) \otimes s_d(y)^*]$ is the kernel of the orthogonal projection onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$, i.e. satisfies (3.2.8). Let $s \in \mathbb{R}\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$, then

$$\int_{y\in\mathcal{X}} \mathbb{E}[s_d(x)\otimes s_d(y)^*]\,(s(y))\,\mathrm{d}V_{\mathcal{X}} = \mathbb{E}\left[s_d(x)\int_{y\in\mathcal{X}} s_d(y)^*(s(y))\,\mathrm{d}V_{\mathcal{X}}\right] = \mathbb{E}[s_d(x)\langle s , s_d\rangle]\,.$$

If s is orthogonal to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ this quantity equals 0. If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ then

$$\begin{aligned}
\mathbb{E}[s_d(x)\langle s , s_d\rangle] &= \mathbb{E}\left[\mathrm{ev}_x^d(s_d)s_d^*(s)\right] \\
&= \mathrm{ev}_x^d\left(\mathbb{E}[s_d\otimes s_d^*]\,(s)\right) \\
&= \mathrm{ev}_x^d(\operatorname{Var}(s_d)\,s) \\
&= \mathrm{ev}_x^d(s) \\
&= s(x),
\end{aligned}$$

since $\operatorname{Var}(s_d) = \mathrm{Id}$. Thus, for any $s \in \mathbb{R}\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$, $\mathbb{E}[s_d(x)\langle s , s_d\rangle]$ is the value at x of the orthogonal projection of s on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Finally, the correlation kernel of $(s_d(x))_{x\in\mathcal{X}}$ satisfies (3.2.8) and equals E_d. □

Remark 3.2.7. If $(s_{1,d}, \dots, s_{N_d,d})$ is any orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we have:

$$E_d : (x, y) \longmapsto \sum_{i=1}^{N_d} s_{i,d}(x) \otimes s_{i,d}(y)^*. \tag{3.2.10}$$

Remark 3.2.8. If $\mathcal{E}$ is the trivial bundle $\mathcal{X} \times \mathbb{C}^r \to \mathcal{X}$ then E_d splits as $E_d = \mathrm{Id} \otimes e_d$, where Id is the identity of $\mathbb{C}^r$ and e_d is the Bergman kernel of $\mathcal{L}^d$. There is no such splitting in general.

Remark 3.2.9. In a complex setting, E_d is also the covariance kernel of the centered Gaussian field associated with a standard complex Gaussian vector in $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

The Bergman kernel also describes the distribution of the derivatives of s_d. Let ∇^d denote any connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$. Then ∇^d induces a dual connection $(\nabla^d)^*$ on $(\mathcal{E} \otimes \mathcal{L}^d)^* \to \mathcal{X}$, which is defined for all $\eta \in \Gamma\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)^*\right)$ by:

$$\forall s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right), \ \forall x \in \mathcal{X}, \qquad d_x \langle s , \eta \rangle = \left\langle \nabla^d_x s , \eta(x) \right\rangle + \left\langle s(x) , (\nabla^d)^*_x \eta \right\rangle, \tag{3.2.11}$$

where $\langle \cdot , \cdot \rangle$ is the duality pairing. Let $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$, then $s^\diamond : x \mapsto s(x)^* = \langle \cdot , s(x) \rangle$ defines a smooth section of $\left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$. Note that we use the notation $s^\diamond$ because s^* already denotes $\langle \cdot , s \rangle$ which is a linear form on $\Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$. We want to understand the relation between $(\nabla^d)^*_x s^\diamond : T_x\mathcal{X} \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*_x$ and $\left(\nabla^d_x s\right)^*$. Recall that $\left(\nabla^d_x s\right)^* = \left\langle \cdot , \nabla^d_x s \right\rangle$, where the inner product is the one on $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T^*_x\mathcal{X}$ induced by h_d and $g_\mathbb{C}$. That is, $\left(\nabla^d_x s\right)^*$ is the adjoint operator of $\nabla^d_x s : T_x\mathcal{X} \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$. In order to get a nice relation, we have to assume that ∇^d is a metric connection, i.e. that:

$$\forall s, t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right), \ \forall x \in \mathcal{X}, \qquad d_x \langle s , t \rangle = \left\langle \nabla^d_x s , t(x) \right\rangle + \left\langle s(x) , \nabla^d_x t \right\rangle. \tag{3.2.12}$$

Lemma 3.2.10. *Let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$, let $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ and let $x \in \mathcal{X}$. Then for all $v \in T_x\mathcal{X}$,*

$$(\nabla^d)^*_x s^\diamond \cdot v = \left(\nabla^d_x s \cdot v\right)^* = v^* \circ \left(\nabla^d_x s\right)^*. \tag{3.2.13}$$

Proof. First, for all $s, t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ and all $x \in \mathcal{X}$,

$$\langle t(x) , s(x) \rangle = \langle t(x) , s(x)^* \rangle = \langle t(x) , s^\diamond(x) \rangle. \tag{3.2.14}$$

Then, by taking the derivative of (3.2.14), we get that for all $s, t \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$, for all $x \in \mathcal{X}$ and $v \in T_x\mathcal{X}$:

$$\left\langle t(x) , \nabla^d_x s \cdot v \right\rangle + \left\langle \nabla^d_x t \cdot v , s(x) \right\rangle = d_x\left(\langle t , s \rangle\right) \cdot v = \left\langle t(x) , (\nabla^d)^*_x s^\diamond \cdot v \right\rangle + \left\langle \nabla^d_x t \cdot v , s^\diamond(x) \right\rangle.$$

The first equality comes from the fact that ∇^d is metric (see (3.2.12)) and the second from the definition of the dual connection (3.2.11). Besides $\left\langle \nabla^d_x t \cdot v , s^\diamond(x) \right\rangle = \left\langle \nabla^d_x t \cdot v , s(x) \right\rangle$, hence for all $s \in \Gamma\left(\mathcal{E} \otimes \mathcal{L}^d\right)$ and all $x \in \mathcal{X}$ we have:

$$\forall v \in T_x\mathcal{X}, \qquad (\nabla^d)^*_x s^\diamond \cdot v = \left(\nabla^d_x s \cdot v\right)^*.$$

Recall that $\left(\nabla^d_x s\right)^*$ is the adjoint of $\nabla^d_x s$. Hence for all $v \in T_x\mathcal{X}$ and all $\zeta \in \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$,

$$\left\langle \zeta , \nabla^d_x s \cdot v \right\rangle = \left\langle \left(\nabla^d_x s\right)^* \zeta , v \right\rangle = v^* \circ \left(\nabla^d_x s\right)^* (\zeta),$$

which proves the second equality in (3.2.13). □

Remark 3.2.11. Conversely, one can show that a connection satisfying the first equality in eq. (3.2.13) for every s, x and v is metric.

From now on, we assume that ∇^d is metric. Then ∇^d induces a natural connection ∇_1^d on $P_1^\star(\mathcal{E}\otimes\mathcal{L}^d) \to \mathcal{X}\times\mathcal{X}$ whose partial derivatives are: ∇^d with respect to the first variable, and the trivial connection with respect to the second. Similarly, $(\nabla^d)^*$ induces a connection ∇_2^d on $P_2^\star\left((\mathcal{E}\otimes\mathcal{L}^d)^*\right)$ and $\nabla_1^d\otimes \mathrm{Id} + \mathrm{Id}\otimes\nabla_2^d$ is a connection on $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$. We denote by ∂_x (resp. ∂_y) its partial derivative with respect to the first (resp. second) variable. By taking partial derivatives in (3.2.9), we get the following.

Corollary 3.2.12. *Let $d\in\mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E}\otimes\mathcal{L}^d$ and let $s_d\sim\mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Then, for all x and $y\in\mathcal{X}$, for all $(v,w)\in T_x\mathcal{X}\times T_y\mathcal{X}$, we have:*

$$\mathrm{Cov}\left(\nabla_x^d s\cdot v, s(y)\right) = \mathbb{E}\left[\left(\nabla_x^d s\cdot v\right)\otimes s(y)^*\right] = \partial_x E_d(x,y)\cdot v, \tag{3.2.15}$$

$$\mathrm{Cov}\left(s(x), \nabla_y^d s\cdot w\right) = \mathbb{E}\left[s(x)\otimes\left(\nabla_y^d s\cdot w\right)^*\right] = \partial_y E_d(x,y)\cdot w, \tag{3.2.16}$$

$$\mathrm{Cov}\left(\nabla_x^d s\cdot v, \nabla_y^d s\cdot w\right) = \mathbb{E}\left[\left(\nabla_x^d s\cdot v\right)\otimes\left(\nabla_y^d s\cdot w\right)^*\right] = \partial_x\partial_y E_d(x,y)\cdot(v,w). \tag{3.2.17}$$

Proof. The first equality of each line is simply the definition of the covariance operator. By applying ∂_x to (3.2.9) we get:

$$\mathbb{E}\left[\left(\nabla_x^d s\right)\otimes s(y)^*\right] = \partial_x E_d(x,y),$$

which proves (3.2.15). We can rewrite (3.2.9) as: $\forall x,y\in\mathcal{X}$, $E_d(x,y) = \mathbb{E}[s(x)\otimes s^\diamond(y)]$. By applying ∂_y to this equality, we get:

$$\mathbb{E}\left[s(x)\otimes\left(\nabla^d\right)_y^* s^\diamond\right] = \partial_y E_d(x,y).$$

Then we apply this operator to $w\in T_y\mathcal{X}$, and we obtain (3.2.16) by Lemma 3.2.10. The proof of (3.2.17) is similar. $\square$

We would like to write that $\partial_y E_d(x,y)$ is $\mathrm{Cov}\left(s(x),\nabla_y^d s\right) = \mathbb{E}\left[s(x)\otimes\left(\nabla_y^d s\right)^*\right]$. Unfortunately, this can not be true since

$$\partial_y E_d(x,y)\in T_y^*\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y^*$$

while

$$\mathbb{E}\left[s(x)\otimes\left(\nabla_y^d s\right)^*\right]\in T_y\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y^*.$$

Let $\partial_y^\sharp E_d(x,y)\in T_y\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y^*$ be defined by:

$$\forall w\in T_y\mathcal{X},\qquad \partial_y^\sharp E_d(x,y)\cdot w^* = \partial_y E_d(x,y)\cdot w. \tag{3.2.18}$$

Similarly, let $\partial_x\partial_y^\sharp E_d(x,y)\in T_x^*\mathcal{X}\otimes T_y\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y^*$ be defined by:

$$\forall (v,w)\in T_x\mathcal{X}\times T_y\mathcal{X},\qquad \partial_x\partial_y^\sharp E_d(x,y)\cdot(v,w^*) = \partial_x\partial_y E_d(x,y)\cdot(v,w). \tag{3.2.19}$$

Then by Lemma 3.2.10 and Corollary 3.2.12, we have the following.

Corollary 3.2.13. *Let $d \in \mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E} \otimes \mathcal{L}^d$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, for all x and $y \in \mathcal{X}$, we have:*

$$\operatorname{Cov}\Big(\nabla_x^d s, s(y)\Big) = \mathbb{E}\Big[\nabla_x^d s \otimes s(y)^*\Big] = \partial_x E_d(x, y), \tag{3.2.20}$$

$$\operatorname{Cov}\Big(s(x), \nabla_y^d s\Big) = \mathbb{E}\Big[s(x) \otimes \Big(\nabla_y^d s\Big)^*\Big] = \partial_y^\sharp E_d(x, y), \tag{3.2.21}$$

$$\operatorname{Cov}\Big(\nabla_x^d s, \nabla_y^d s\Big) = \mathbb{E}\Big[\nabla_x^d s \otimes \Big(\nabla_y^d s\Big)^*\Big] = \partial_x \partial_y^\sharp E_d(x, y). \tag{3.2.22}$$

3.3 Estimates for the Bergman kernel

The goal of this section is to recall the estimates we need for the Bergman kernel. Most of what follows can be found in [MM07], with small additions from [MM13] and [MM15]. The first to use this kind of estimates in a random geometry context were Shiffman and Zelditch [SZ99]. They used the estimates from [Zel98] for the related Szegö kernel (see also [BSZ00, SZ08]). Catlin [Cat99] proved similar estimates for the Bergman kernel independently.

In order to state the near-diagonal estimates for the Bergman kernel, we first need to choose preferred charts on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ around any point in M. This is done in Section 3.3.1. Unlike our main reference [MM07], we are only concerned with a neighborhood of the real locus of $\mathcal{X}$, but we need to check that these charts are well-behaved with respect to the real structures. Sections 3.3.2, 3.3.3 and 3.3.4 state respectively near-diagonal, diagonal and far off-diagonal estimates for E_d.

3.3.1 Real normal trivialization

In this section, we define preferred local trivializations for $\mathcal{E}$ and $\mathcal{L}$ around any point in M. We also prove that these trivializations are compatible with the real and metric structures.

Let $R > 0$ be such that the injectivity radius of $\mathcal{X}$ is larger than $2R$. Let $x_0 \in M$, then the exponential map $\exp_{x_0} : T_{x_0}\mathcal{X} \to \mathcal{X}$ at x_0 is a diffeomorphism from the ball $B_{T_{x_0}\mathcal{X}}(0, 2R) \subset T_{x_0}\mathcal{X}$ to the geodesic ball $B_\mathcal{X}(x_0, 2R) \subset \mathcal{X}$. Note that this diffeomorphism is not biholomorphic in general.

Notation 3.3.1. Here and in the sequel, we always denote by $B_A(a, R)$ the open ball of center a and radius $R > 0$ in the metric space A.

Since $c_\mathcal{X}$ is an isometry (see Sect. 3.2.1), we have that $c_\mathcal{X} \circ \exp_{x_0} = \exp_{x_0} \circ d_{x_0}c_\mathcal{X}$. Then $\exp_{x_0}$ sends $T_{x_0}M = \ker(d_{x_0}c_\mathcal{X} - \mathrm{Id})$ to M and agrees on $T_{x_0}M$ with the exponential map at x_0 in (M, g). By restriction, we get a diffeomorphism from $B_{T_{x_0}M}(0, 2R) \subset T_{x_0}M$ to the geodesic ball $B_M(x_0, 2R) \subset M$. Moreover, on $B_{T_{x_0}\mathcal{X}}(0, 2R)$ we have:

$$d_{x_0}c_\mathcal{X} = (\exp_{x_0})^{-1} \circ c_\mathcal{X} \circ \exp_{x_0}. \tag{3.3.1}$$

We say that $\exp_{x_0}$ defines a *real normal chart* about x_0.

Since $i \cdot T_{x_0}M = \ker(d_{x_0}c_\mathcal{X} + \mathrm{Id})$, we have $T_{x_0}\mathcal{X} = T_{x_0}M \oplus i \cdot T_{x_0}M$. Note that $T_{x_0}M$ and $i \cdot T_{x_0}M$ are orthogonal for g_{x_0}, since these are distinct eigenspaces of an isometric involution. Moreover, we know from Sect. 3.2.1 that $c_\mathcal{X}^\star g_\mathbb{C} = \overline{g_\mathbb{C}}$. This implies that $(g_\mathbb{C})_{x_0}$ takes real values on $T_{x_0}M \times T_{x_0}M$, i.e. the restrictions to $T_{x_0}M$ of $(g_\mathbb{C})_{x_0}$ and g_{x_0} are equal. Thus, $(g_\mathbb{C})_{x_0}$ is the sesquilinear extension of g_{x_0} restricted to $T_{x_0}M$. Let $\mathcal{I}$ be an isometry from $T_{x_0}M$ to $\mathbb{R}^n$ with its standard Euclidean structure, $\mathcal{I}$ extends as a $\mathbb{C}$-linear isometry $\mathcal{I}_\mathbb{C} : T_{x_0}\mathcal{X} \to \mathbb{C}^n$, such that $\mathcal{I}_\mathbb{C} \circ d_{x_0}c_\mathcal{X} \circ \mathcal{I}_\mathbb{C}^{-1}$ is the complex conjugation in $\mathbb{C}^n$. Thus, $\exp_{x_0} \circ \mathcal{I}_\mathbb{C}^{-1} : B_{\mathbb{C}^n}(0, 2R) \to B_\mathcal{X}(x_0, 2R)$ defines normal coordinates that induce normal coordinates $B_{\mathbb{R}^n}(0, 2R) \to B_M(x_0, 2R)$ and such that $\mathcal{I}_\mathbb{C} \circ (\exp_{x_0})^{-1} \circ c_\mathcal{X} \circ \exp_{x_0} \circ \mathcal{I}_\mathbb{C}^{-1}$

is the complex conjugation in $\mathbb{C}^n$. Such coordinates are called *real normal coordinates* about x_0.

We can now trivialize $\mathcal{E}$ over $B_{\mathcal{X}}(x_0, 2R)$. Let $\nabla^{\mathcal{E}}$ denote the Chern connection of $\mathcal{E}$. We identify the fiber at $\exp_{x_0}(z) \in B_{\mathcal{X}}(x_0, 2R)$ with $\mathcal{E}_{x_0}$, by parallel transport with respect to $\nabla^{\mathcal{E}}$ along the geodesic from x_0 to $\exp_{x_0}(z)$, defined by $t \mapsto \exp_{x_0}(tz)$ from $[0,1]$ to $\mathcal{X}$ (cf. [MM07, sect. 1.6] and [MM13]). This defines a bundle map

$$\varphi_{x_0} : B_{T_{x_0}\mathcal{X}}(0, 2R) \times \mathcal{E}_{x_0} \to \mathcal{E}_{/B_{\mathcal{X}}(x_0, 2R)}$$

that covers $\exp_{x_0}$. We say that φ_{x_0} is the *real normal trivialization* of $\mathcal{E}$ over $B_{\mathcal{X}}(x_0, 2R)$.

Since $x_0 \in M$, $c_{\mathcal{E}}(\mathcal{E}_{x_0}) = \mathcal{E}_{x_0}$ and we denote by $c_{\mathcal{E},x_0}$ the restriction of $c_{\mathcal{E}}$ to $\mathcal{E}_{x_0}$. Then $(d_{x_0}c_{\mathcal{X}}, c_{\mathcal{E},x_0})$ is a real structure on $B_{T_{x_0}\mathcal{X}}(0, 2R) \times \mathcal{E}_{x_0}$ compatible with the real structure on $B_{T_{x_0}\mathcal{X}}(0, 2R)$. We want to check that φ_{x_0} is well-behaved with respect to the real structures, i.e. that for all $z \in B_{T_{x_0}\mathcal{X}}(0, 2R)$ and $\zeta^0 \in \mathcal{E}_{x_0}$,

$$c_{\mathcal{E}}(\varphi_{x_0}(z, \zeta^0)) = \varphi_{x_0}\left(d_{x_0}c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right). \tag{3.3.2}$$

This will be a consequence of Lemma 3.3.4 below.

Definition 3.3.2. Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle equipped with compatible real structures $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ and let ∇ be a connection on $\mathcal{E}$, we say that ∇ is a *real connection* if for every section $s \in \Gamma(\mathcal{E})$ we have:

$$\forall x \in \mathcal{X}, \qquad \nabla_x\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) = c_{\mathcal{E}} \circ \nabla_{c_{\mathcal{X}}(x)} s \circ d_x c_{\mathcal{X}}.$$

Remark 3.3.3. Let $x \in M$, $v \in T_xM$ and $s \in \mathbb{R}\Gamma(\mathcal{E})$. If ∇ is a real connection on $\mathcal{E}$, then $\nabla_x s \cdot v \in \mathbb{R}\mathcal{E}_x$. Indeed,

$$\nabla_x s \cdot v = \nabla_{c_{\mathcal{X}}(x)} s \circ d_x c_{\mathcal{X}} \cdot v = c_{\mathcal{E}}\left(\nabla_x\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \cdot v\right) = c_{\mathcal{E}}\left(\nabla_x s \cdot v\right).$$

Lemma 3.3.4. *Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle equipped with compatible real structures $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ and a real Hermitian metric $h_{\mathcal{E}}$. Then, the Chern connection $\nabla^{\mathcal{E}}$ of $\mathcal{E}$ is real.*

Proof. Since $c_{\mathcal{E}}$ and $c_{\mathcal{X}}$ are involutions and $(d_x c_{\mathcal{X}})^{-1} = d_{c_{\mathcal{X}}(x)} c_{\mathcal{X}}$, we need to check that

$$\forall s \in \Gamma(\mathcal{E}), \forall x \in \mathcal{X} \qquad \nabla^{\mathcal{E}}_x s = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \circ d_x c_{\mathcal{X}}. \tag{3.3.3}$$

Let $\widetilde{\nabla}$ be defined by $\widetilde{\nabla}_x s = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \circ d_x c_{\mathcal{X}}$, for all $s \in \Gamma(\mathcal{E})$ and $x \in \mathcal{X}$. Then $\widetilde{\nabla}$ is a connection on $\mathcal{E}$ and it is enough to check that it is compatible with both the metric and the complex structure. Indeed, in this case $\widetilde{\nabla} = \nabla^{\mathcal{E}}$ by unicity of the Chern connection, which proves (3.3.3).

Let us check that $\widetilde{\nabla}$ satisfies Leibniz' rule. Let $s \in \Gamma(\mathcal{E})$ and $f : \mathcal{X} \to \mathbb{C}$. We have:

$$\begin{aligned}
\widetilde{\nabla}_x(fs) &= c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}\left((\overline{f \circ c_{\mathcal{X}}})(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= c_{\mathcal{E}} \circ \left(\overline{f(x)} \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) + d_{c_{\mathcal{X}}(x)}(\overline{f \circ c_{\mathcal{X}}}) \otimes c_{\mathcal{E}}(s(x))\right) \circ d_x c_{\mathcal{X}} \\
&= f(x) \widetilde{\nabla}_x s + d_x f \otimes s(x).
\end{aligned}$$

Since $\nabla^{\mathcal{E}}$ is the Chern connection, its anti-holomorphic part is $\overline{\partial}^{\mathcal{E}}$. Then, $d_x c_{\mathcal{X}}$ and $c_{\mathcal{E}}$ being anti-linear (resp. fiberwise), the anti-linear part of $\widetilde{\nabla}_x s$ equals:

$$c_{\mathcal{E}} \circ \overline{\partial}^{\mathcal{E}}_{c_{\mathcal{X}}(x)}\left(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}\right) \circ d_x c_{\mathcal{X}}.$$

By computing in a local holomorphic frame, one can check that:

$$\forall s \in \Gamma(\mathcal{E}), \forall x \in \mathcal{X}, \qquad c_{\mathcal{E}} \circ \overline{\partial}^{\mathcal{E}}_{c_{\mathcal{X}}(x)} (c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}) \circ d_x c_{\mathcal{X}} = \overline{\partial}^{\mathcal{E}}_x s.$$

Thus, $\widetilde{\nabla}$ is compatible with the complex structure. Finally, we check the compatibility with the metric structure. Let $s, t \in \Gamma(\mathcal{E})$ and $x \in \mathcal{X}$, since $h_{\mathcal{E}} = c_{\mathcal{E}}^{\star}(\overline{h_{\mathcal{E}}})$ we have:

$$\begin{aligned}
d_x(h_{\mathcal{E}}(s,t)) = d_x \left(\overline{h_{\mathcal{E}}}(c_{\mathcal{E}} \circ s, c_{\mathcal{E}} \circ t)\right) &= d_{c_{\mathcal{X}}(x)} \left(\overline{h_{\mathcal{E}}}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}, c_{\mathcal{E}} \circ t \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= \overline{h_{\mathcal{E}}} \left(\nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}), c_{\mathcal{E}}(t(x))\right) \circ d_x c_{\mathcal{X}} \\
&\qquad + \overline{h_{\mathcal{E}}} \left(c_{\mathcal{E}}(s(x)), \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ t \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= h_{\mathcal{E}} \left(c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}}), t(x)\right) \circ d_x c_{\mathcal{X}} \\
&\qquad + h_{\mathcal{E}} \left(s(x), c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(x)}(c_{\mathcal{E}} \circ s \circ c_{\mathcal{X}})\right) \circ d_x c_{\mathcal{X}} \\
&= h_{\mathcal{E}} \left(\widetilde{\nabla}_x s, t(x)\right) + h_{\mathcal{E}} \left(s(x), \widetilde{\nabla}_x t\right).
\end{aligned}$$

□

Let us now prove (3.3.2). Let $z \in B_{T_{x_0}\mathcal{X}}(0, 2R)$, let $\zeta^0 \in \mathcal{E}_{x_0}$ and let $\zeta : B_{\mathcal{X}}(x_0, 2R) \to \mathcal{E}$ be the section defined by $\zeta : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta^0\right)$. We denote by $\gamma : [0,1] \mapsto \mathcal{X}$ the geodesic $t \mapsto \exp_{x_0}(tz)$. We have for all $t \in [0,1]$, $\zeta(\gamma(t)) = \varphi_{x_0}(tz, \zeta^0)$ and, by the definition of φ_{x_0}, we have:

$$\forall t \in [0,1], \qquad \nabla^{\mathcal{E}}_{\gamma(t)} \zeta \cdot \gamma'(t) = 0. \tag{3.3.4}$$

Let us denote $\widetilde{\zeta} = c_{\mathcal{E}} \circ \zeta \circ c_{\mathcal{X}}$ and $\overline{\gamma} = c_{\mathcal{X}} \circ \gamma$. Since $\nabla^{\mathcal{E}}$ is real, (3.3.4) implies that for all $t \in [0,1]$,

$$\nabla^{\mathcal{E}}_{\overline{\gamma}(t)} \widetilde{\zeta} \cdot \overline{\gamma}'(t) = \nabla^{\mathcal{E}}_{c_{\mathcal{X}}(\gamma(t))} \widetilde{\zeta} \circ d_{\gamma(t)}(c_{\mathcal{X}}) \cdot \gamma'(t) = c_{\mathcal{E}} \circ \nabla^{\mathcal{E}}_{\gamma(t)} \zeta \cdot \gamma'(t) = 0. \tag{3.3.5}$$

Since $c_{\mathcal{X}}$ is an isometry, $\overline{\gamma}$ is a geodesic. More precisely, $\overline{\gamma} : t \mapsto \exp_{x_0}(t d_{x_0} c_{\mathcal{X}} \cdot z)$. Besides, $\widetilde{\zeta}(x_0) = c_{\mathcal{E}}(\zeta(x_0)) = c_{\mathcal{E},x_0}(\zeta^0)$. Then by (3.3.5), for all $t \in [0,1]$,

$$\varphi_{x_0}\left(t d_{x_0} c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right) = \varphi_{x_0}\left(t d_{x_0} c_{\mathcal{X}} \cdot z, \widetilde{\zeta}(x_0)\right) = \widetilde{\zeta}(\overline{\gamma}(t)).$$

Finally, we get (3.3.2) for $t = 1$:

$$\varphi_{x_0}\left(d_{x_0} c_{\mathcal{X}} \cdot z, c_{\mathcal{E},x_0}(\zeta^0)\right) = \widetilde{\zeta}(\overline{\gamma}(1)) = c_{\mathcal{E}}(\zeta(\gamma(1))) = c_{\mathcal{E}}\left(\varphi_{x_0}(z, \zeta^0)\right).$$

Recall that $\mathbb{R}\mathcal{E}$ is the set of fixed points of $c_{\mathcal{E}}$. Then $\mathbb{R}\mathcal{E}$ is naturally a rank r real vector bundle over M, as a subbundle of $\mathcal{E}_{/M}$. Let $\zeta^0 \in \mathbb{R}\mathcal{E}_{x_0}$, and $\zeta : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta^0\right)$ then, for all $x \in B_{\mathcal{X}}(x_0, 2R)$,

$$\begin{aligned}
c_{\mathcal{E}} \circ \zeta \circ c_{\mathcal{X}}(x) &= c_{\mathcal{E}} \circ \varphi_{x_0}\left((\exp_{x_0})^{-1}(c_{\mathcal{X}}(x)), \zeta^0\right) \\
&= c_{\mathcal{E}} \circ \varphi_{x_0}\left(d_{x_0} c_{\mathcal{X}} \circ (\exp_{x_0})^{-1}(x), \zeta^0\right) \\
&= \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), c_{\mathcal{E},x_0}(\zeta^0)\right) \\
&= \zeta(x).
\end{aligned}$$

Hence, ζ is a real local section of $\mathcal{E}$ and in particular, $\forall x \in M$, $\zeta(x) \in \mathbb{R}\mathcal{E}_x$. This shows that φ_{x_0} induces, by restriction, a bundle map $B_{T_{x_0}M}(0, 2R) \times \mathbb{R}\mathcal{E}_{x_0} \to \mathbb{R}\mathcal{E}_{/B_M(x_0,2R)}$ that covers the restriction of $\exp_{x_0}$ to $B_{T_{x_0}M}(0, 2R)$.

Let $(\zeta_1^0, \dots, \zeta_r^0)$ be an orthonormal basis of $\mathbb{R}\mathcal{E}_{x_0}$. Since $\mathbb{R}\mathcal{E}_{x_0} = \ker(c_{\mathcal{E},x_0} - \mathrm{Id})$ and $c_{\mathcal{E},x_0}$ is $\mathbb{C}$-anti-linear, we have $\mathcal{E}_{x_0} = \mathbb{R}\mathcal{E}_{x_0} \oplus i \cdot \mathbb{R}\mathcal{E}_{x_0}$. Moreover, since $h_{\mathcal{E},x_0}$ and $c_{\mathcal{E},x_0}$ are compatible, $(\zeta_1^0, \dots, \zeta_r^0)$ is also an orthonormal basis of $\mathcal{E}_{x_0}$. Let $i \in \{1, \dots, r\}$, we denote by $\zeta_i : B_{\mathcal{X}}(x_0, 2R) \to \mathcal{E}$ the real local section defined by:

$$\zeta_i : x \mapsto \varphi_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta_i^0\right).$$

Then, for every $x \in B_{\mathcal{X}}(0, 2R)$, $(\zeta_1(x), \dots, \zeta_r(x))$ is an orthonormal basis of $\mathcal{E}_x$. Indeed, the sections ζ_i are obtained by parallel transport for $\nabla^{\mathcal{E}}$ along geodesics starting at x_0, and $\nabla^{\mathcal{E}}$ is compatible with $h_{\mathcal{E}}$. Hence, for all i and $j \in \{1, \dots, r\}$, for all $z \in B_{\mathcal{X}}(x_0, 2R)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(h_{\mathcal{E}}(\zeta_i(\exp_{x_0}(tz)), \zeta_j(\exp_{x_0}(tz)))\right) &= h_{\mathcal{E}}\left(\nabla^{\mathcal{E}}_{\exp_{x_0}(tz)}\zeta_i \circ d_{tz}\exp_{x_0}\cdot z, \zeta_j(\exp_{x_0}(tz)))\right) \\ &\quad + h_{\mathcal{E}}\left(\zeta_i(\exp_{x_0}(tz))), \nabla^{\mathcal{E}}_{\exp_{x_0}(tz)}\zeta_j \circ d_{tz}\exp_{x_0}\cdot z\right) = 0.\end{aligned}$$

The function $x \mapsto h_{\mathcal{E}}(\zeta_i(x), \zeta_j(x))$ is then constant along geodesics starting at x_0, hence on $B_{\mathcal{X}}(x_0, 2R)$. Since $(h_{\mathcal{E}}(\zeta_i(x), \zeta_j(x)))_{1 \leqslant i,j \leqslant r}$ is the identity matrix of size r at x_0, $(\zeta_1, \dots, \zeta_r)$ is a smooth unitary frame for $\mathcal{E}$ over $B_{\mathcal{X}}(0, 2R)$. In particular, this shows that the real normal trivialization φ_{x_0} is unitary. Since the ζ_i are real, $(\zeta_1(x), \dots, \zeta_r(x))$ is an orthonormal basis of $\mathbb{R}\mathcal{E}_x$ for all $x \in M$. Hence $(\zeta_1, \dots, \zeta_r)$ is also a smooth orthogonal frame for $\mathbb{R}\mathcal{E}$ over $B_M(0, 2R)$. We say that $(\zeta_1, \dots, \zeta_r)$ is a local *real unitary frame*.

Similarly, let φ'_{x_0} denote the real normal trivialization of $\mathcal{L}$ over $B_{\mathcal{X}}(x_0, 2R)$. Then any unit vector $\zeta_0^0 \in \mathbb{R}\mathcal{L}_{x_0}$ defines a local real unitary frame ζ_0 for $\mathcal{L}$:

$$\zeta_0 : x \mapsto \varphi'_{x_0}\left((\exp_{x_0})^{-1}(x), \zeta_0^0\right).$$

Then, for any $d \in \mathbb{N}$, φ_{x_0} and φ'_{x_0} induce a trivialization $\varphi_{x_0} \otimes (\varphi'_{x_0})^d$ of $\mathcal{E} \otimes \mathcal{L}^d$. This trivialization is the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x_0, 2R)$, i.e. it is obtained by parallel transport along geodesics starting at x_0 for the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$. Moreover, a local real unitary frame for $\mathcal{E} \otimes \mathcal{L}^d$ is given by $(\zeta_1 \otimes \zeta_0^d, \dots, \zeta_r \otimes \zeta_0^d)$.

3.3.2 Near-diagonal estimates

We can now state the near-diagonal estimates of Ma and Marinescu for the Bergman kernel. In the sequel, we fix some $R > 0$ such that $2R$ is smaller than the injectivity radius of $\mathcal{X}$. Let $x \in M$, we have a natural real normal chart

$$\exp_x \times \exp_x : B_{T_x\mathcal{X}}(0, 2R) \times B_{T_x\mathcal{X}}(0, 2R) \to B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R).$$

Moreover, the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x, 2R)$ (see Section 3.3.1) induces a trivialization

$$B_{T_x\mathcal{X}}(0, 2R) \times B_{T_x\mathcal{X}}(0, 2R) \times \operatorname{End}\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right) \simeq \left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*_{/B_{\mathcal{X}}(x,2R) \times B_{\mathcal{X}}(x,2R)}$$

that covers $\exp_x \times \exp_x$. This trivialization coincides with the real normal trivialization of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ over $B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R)$.

Recall that $\mathrm{d}V_{\mathcal{X}}$ denotes the Riemannian measure on $\mathcal{X}$. When we read this measure in the real normal chart $\exp_x$, it admits a density $\kappa : B_{T_x\mathcal{X}}(0, 2R) \to \mathbb{R}_+$ with respect to the normalized Lebesgue measure of $(T_x\mathcal{X}, g_x)$. More precisely, we have $\kappa(z) = \sqrt{\det(g_{ij}(z))}$ where $(g_{ij}(z))$ is the matrix of $((\exp_x)^\star g)_z$, read in any real orthonormal basis of $(T_x\mathcal{X}, g_x)$. Since we use normal coordinates and $\mathcal{X}$ is compact, we have

$$\kappa(z) = 1 + O\left(\|z\|^2\right) \tag{3.3.6}$$

where $\|\cdot\|$ is induced by g_x and the estimate $O\left(\|z\|^2\right)$ does not depend on x.

Similarly, on the real locus (M, g), $|\mathrm{d}V_M|$ admits a density, in the real normal chart $\exp_x$, with respect to the normalized Lebesgue measure on (T_xM, g_x). This density is:

$$z \longmapsto \det\left(((\exp_x^\star g)_z)_{/T_xM}\right)^{\frac{1}{2}}, \tag{3.3.7}$$

from $B_{T_xM}(0, 2R)$ to $\mathbb{R}_+$. As we already explained in Sect. 3.3.1, on the real locus, $g_{\mathbb{C}}$ is the sesquilinear extension of the restriction of g to TM. Hence, for all $z \in B_{T_xM}(0, 2R)$ we have:

$$\det\left(((\exp_x^\star g)_z)_{/T_xM}\right)^2 = \det\left((\exp_x^\star g)_z\right),$$

which means that the density of $|\mathrm{d}V_M|$ in the chart $\exp_x$ is $\sqrt{\kappa} : B_{T_xM}(0, 2R) \to \mathbb{R}_+$.

The following result gives the asymptotic of the Bergman kernel E_d (see Sect. 3.2.3) and its derivatives, read in the real normal trivialization about x of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$. It was first established by Dai, Liu and Ma in [DLM06, thm. 4.18'].

Theorem 3.3.5 (Ma–Marinescu)**.** *There exists $C' > 0$ such that, for any $p \in \mathbb{N}$, there exists C_p such that $\forall k \in \{0, \dots, p\}$, $\forall d \in \mathbb{N}$, $\forall z, w \in B_{T_x\mathcal{X}}(0, R)$,*

$$\left\| D^k_{(z,w)} \left(E_d(z, w) - \left(\frac{d}{\pi}\right)^n \frac{\exp\left(-\frac{d}{2}\left(\|z\|^2 + \|w\|^2 - 2\langle z, w\rangle\right)\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} \right) \right\|$$
$$\leqslant C_p d^{n+\frac{p}{2}-1} \left(1 + \sqrt{d}(\|z\| + \|w\|)\right)^{2n+6+p} \exp\left(-C'\sqrt{d}\,\|z - w\|\right) + O\big(d^{-\infty}\big),$$

where:

- $D^k_{(z,w)}$ *is the k-th differential at (z, w) for a map $T_x\mathcal{X} \times T_x\mathcal{X} \to \mathrm{End}\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right)$,*
- *the Hermitian inner product $\langle \cdot , \cdot \rangle$ comes from the Hermitian metric $(g_{\mathbb{C}})_x$,*
- *the norm $\|\cdot\|$ on $T_x\mathcal{X}$ is induced by g_x (or equivalently $\langle \cdot , \cdot \rangle$),*
- *the norm $\|\cdot\|$ on $(T_x^*\mathcal{X})^{\otimes q} \otimes \mathrm{End}\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right)$ is induced by g_x and $(h_d)_x$.*

Moreover, the constants C_p and C' do not depend on x. The notation $O(d^{-\infty})$ means that, for any $l \in \mathbb{N}$, this term is $O\big(d^{-l}\big)$ with a constant that does not depend on x, z, w or d.

Proof. This is a weak version of [MM07, thm. 4.2.1], with $k = 1$ and $m' = 0$ in the notations of [MM07]. We used the fact that $\mathcal{F}_0$ in [MM07] is given by:

$$\mathcal{F}_0(z, w) = \frac{1}{\pi^n} \exp\left(-\frac{1}{2}\left(\|z\|^2 + \|w\|^2 - 2\langle z, w\rangle\right)\right) \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x},$$

(compare (4.1.84), (4.1.85) et (4.1.92) pp. 191–192 and (5.1.18) p. 46 in [MM07]) and $\mathcal{F}_1 = 0$. See [MM07, Rem. 1.4.26] and [MM13]. $\square$

Remark 3.3.6. Note that our formula differs from the one in [MM07, MM13] by a factor π in the exponential. This comes from different normalizations of the Kähler form ω.

We are only interested in the behavior of E_d at points of the real locus, hence we restrict our focus to points in M and derivatives in real directions. Similarly, for $x, y \in M$, $E_d(x, y)$ restricts to an element of $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_y^*$, still denoted by $E_d(x, y)$. Note that we can recover the original $E_d(x, y) : \left(\mathcal{E} \otimes \mathcal{L}^d\right)_y \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ from its restriction by $\mathbb{C}$-linear extension.

First, we need to know the behavior of E_d and its derivatives up to order 1 in each variable in a neighborhood of the diagonal in $M \times M$.

Corollary 3.3.7. *There exist C and $C' > 0$, not depending on x, such that $\forall k \in \{0,1,2\}$, $\forall d \in \mathbb{N}$, $\forall z, w \in B_{T_xM}(0,R)$,*

$$\left\| D^k_{(z,w)} \left(E_d(z,w) - \left(\frac{d}{\pi}\right)^n \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right) \right\| \\ \leqslant C d^{n+\frac{k}{2}-1} \left(1+\sqrt{d}(\|z\|+\|w\|)\right)^{2n+8} \exp\left(-C'\sqrt{d}\,\|z-w\|\right) + O\left(d^{-\infty}\right),$$

where D^k is the k-th differential for a map from $T_xM \times T_xM$ to $\operatorname{End}\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$*, the norm on T_xM is induced by g_x and the norm on $(T^*_xM)^{\otimes q} \otimes$* $\operatorname{End}\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$ *is induced by g_x and $(h_d)_x$.*

Proof. We apply Theorem. 3.3.5 for $p = k \in \{0,1,2\}$ and set $C = \max(C_0, C_1, C_2)$. Then we restrict everything to the real locus. □

3.3.3 Diagonal estimates

In this section, we deduce diagonal estimates for E_d and its derivatives from Thm. 3.3.5. Let $x \in M$, then the usual differential for maps from $T_x\mathcal{X}$ to $(\mathcal{E}\otimes\mathcal{L}^d)_x$ defines a local trivial connection $\widetilde{\nabla}^d$ on $(\mathcal{E}\otimes\mathcal{L}^d)_{/B_\mathcal{X}(0,2R)}$, via the real normal trivialization. Since this trivialization is well-behaved with respect to both the metric and the real structure (cf. Sect. 3.3.1), $\widetilde{\nabla}^d$ is metric and real. By a partition of unity argument, there exists a real metric connection ∇^d on $\mathcal{E}\otimes\mathcal{L}^d$ such that ∇^d agrees with $\widetilde{\nabla}^d$ on $B_\mathcal{X}(0,R)$. In the remainder of this section, we use this connection ∇^d, and the induced connection on $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$, so that the connection is trivial in the real normal trivialization about x.

Recall that $\partial^\sharp_y E_d$ and $\partial_x\partial^\sharp_y E_d$ are defined by (3.2.18) and (3.2.19) respectively.

Corollary 3.3.8. *Let $x \in M$, let ∇^d be a real metric connection that is trivial over $B_{T_x\mathcal{X}}(0,R)$ in the real normal trivialization about x. Let $\partial^\sharp_y$ and ∂_x denote the associated partial derivatives for sections of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$, then we have the following estimates as $d \to +\infty$.*

$$E_d(x,x) = \frac{d^n}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{n-1}\right), \tag{3.3.8}$$

$$\partial_x E_d(x,x) = O\left(d^{n-\frac{1}{2}}\right), \tag{3.3.9}$$

$$\partial^\sharp_y E_d(x,x) = O\left(d^{n-\frac{1}{2}}\right), \tag{3.3.10}$$

$$\partial_x\partial^\sharp_y E_d(x,x) = \frac{d^{n+1}}{\pi^n} \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \otimes \operatorname{Id}_{T^*_xM} + O(d^n). \tag{3.3.11}$$

Moreover the error terms do not depend on x.

Proof. Let $x \in M$ and let us choose an orthonormal basis of T_xM. We denote the corresponding coordinates on $T_xM \times T_xM$ by $(z_1, \dots, z_n, w_1, \dots, w_n)$ and by ∂_{z_i} and ∂_{w_j} the associated partial derivatives. Let us compute the partial derivatives of E_d read in the real normal trivialization of $\left(\mathcal{E}\otimes\mathcal{L}^d\right)\boxtimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$ about (x,x). By Cor. 3.3.7, we only need to compute the partial derivatives at $(0,0)$ of

$$\xi_d : (z,w) \mapsto \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \tag{3.3.12}$$

for any $d \in \mathbb{N}$. For all i and $j \in \{1, \dots, n\}$ and all $(z, w) \in B_{T_xM}(0, R)$ we have:

$$\partial_{z_i}\xi_d(z,w) = \left(-d(z_i - w_i) - \frac{1}{2}\frac{\partial_{z_i}\kappa(z)}{\kappa(z)}\right)\frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}}, \tag{3.3.13}$$

$$\partial_{w_j}\xi_d(z,w) = \left(d(z_j - w_j) - \frac{1}{2}\frac{\partial_{w_j}\kappa(w)}{\kappa(w)}\right)\frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}} \tag{3.3.14}$$

and

$$\partial_{z_i}\partial_{w_j}\xi_d(z,w) = \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(z)}\sqrt{\kappa(w)}}\times$$
$$\left(d\delta_{ij} - d^2(z_i - w_i)(z_j - w_j) - \frac{d(z_j - w_j)}{2}\frac{\partial_{z_i}\kappa(z)}{\kappa(z)} + \frac{d(z_i - w_i)}{2}\frac{\partial_{w_j}\kappa(w)}{\kappa(w)}\right), \tag{3.3.15}$$

where δ_{ij} equals 1 if $i = j$ and 0 otherwise. Recall that, by (3.3.6), $\kappa(0) = 1$ and the partial derivatives of κ vanish at the origin. Then evaluating the above expressions at $(0, 0)$ gives:

$$\xi_d(0,0) = 1, \qquad \partial_{z_i}\xi_d(0,0) = 0 = \partial_{w_j}\xi_d(0,0) \qquad \text{and} \qquad \partial_{z_i}\partial_{w_j}\xi_d(0,0) = \delta_{ij}d.$$

By Cor. 3.3.7, we have the following estimates for the partial derivatives of E_d read in the real normal trivialization about x: for all $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} E_d(0,0) &= \frac{d^n}{\pi^n}\operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{n-1}\right), & \partial_{w_j}E_d(0,0) &= O\left(d^{n-\frac{1}{2}}\right), \\ \partial_{z_i}\partial_{w_j}E_d(0,0) &= \delta_{ij}\frac{d^{n+1}}{\pi^n}\operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^n\right), & \partial_{z_i}E_d(0,0) &= O\left(d^{n-\frac{1}{2}}\right). \end{aligned} \tag{3.3.16}$$

Moreover these estimates are uniform in $x \in M$. Equations (3.3.8), (3.3.9), (3.3.10) and (3.3.11) are coordinate-free versions of these statements. □

3.3.4 Far off-diagonal estimates

Finally, we will use the fact that the Bergman kernel and its derivatives decrease fast enough outside of the diagonal. In this section we recall the far off-diagonal estimates of [MM15, thm. 5], see also [MM07, prop. 4.1.5].

Let $d \in \mathbb{N}$ and let S be a smooth section of $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*$. Let $x, y \in M$, we denote by $\|S(x,y)\|_{\mathcal{C}^k}$ the maximum of the norms of S and its derivatives of order at most k at the point (x, y). The derivatives of S are computed with respect to the connection induced by the Chern connection of $\mathcal{E}\otimes\mathcal{L}^d$ and the Levi–Civita connection on M. The norms of the derivatives are the one induced by h_d and g.

Theorem 3.3.9 (Ma–Marinescu)**.** *There exist $C' > 0$ and $d_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that $\forall d \geqslant d_0$, $\forall x, y \in M$*

$$\|E_d(x,y)\|_{\mathcal{C}^k} \leqslant C_k d^{n+\frac{k}{2}}\exp\left(-C'\sqrt{d}\,\rho_g(x,y)\right),$$

where $\rho_g(\cdot,\cdot)$ denotes the geodesic distance in (M, g).

Proof. This is the first part of [MM15, thm. 5], where we only considered the restriction of E_d and its derivatives to M. Note that the Levi–Civita connection on M is the restriction of the Levi–Civita connection on $\mathcal{X}$. Hence the norm $\|\cdot\|_{\mathcal{C}^k}$, such as we defined it, is smaller than the one used in [MM15]. □

3.4 Proof of Theorem 3.1.6

In this section, we prove Theorem 3.1.6. Recall that $\mathcal{X}$ is a compact Kähler manifold of dimension $n \geqslant 2$ defined over the reals and that M denotes its real locus, assumed to be non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n-1\}$ real Hermitian vector bundle and $\mathcal{L} \to \mathcal{X}$ be a real Hermitian line bundle whose curvature form is ω, the Kähler form of $\mathcal{X}$. We assume that $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. For all $d \in \mathbb{N}$, E_d denotes the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Finally, s_d denotes a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, which real zero set is denoted by Z_d, and $|\mathrm{d}V_d|$ is the measure of integration over Z_d.

3.4.1 The Kac–Rice formula

The first step in our proof of Thm. 3.1.6 is to prove a version of the Kac–Rice formula adapted to our problem. This is the goal of this section. First, we recall the Kac–Rice formula we used in [Let16a] to compute the expectation of $\mathrm{Vol}\,(Z_d)$ (Thm. 3.4.1). Then we prove a Kac–Rice formula adapted to the computation of the covariance (Thm. 3.4.4), compare [AW09, thm. 6.3] and [TA07, chap. 11.5].

Let $L : V \to V'$ be a linear map between two Euclidean spaces, recall that we denote by $\left|\det^{\perp}(L)\right|$ its Jacobian (cf. Def. 3.1.3). Since LL^* is a semi-positive symmetric endomorphism of V', $\det(LL^*) \geqslant 0$ and $\left|\det^{\perp}(L)\right|$ is well-defined. The range of L^* is $\ker(L)^{\perp}$, hence $\ker(LL^*) = \ker(L^*) = L(V)^{\perp}$. Thus $\left|\det^{\perp}(L)\right| > 0$ if and only if LL^* is injective, that is if and only if L is surjective. In fact, if L is surjective, let A be the matrix of the restriction of L to $\ker(L)^{\perp}$ in any orthonormal basis of $\ker(L)^{\perp}$ and V', then we have:

$$\left|\det^{\perp}(L)\right| = \sqrt{\det\left(AA^{\mathrm{t}}\right)} = \left|\det(A)\right|.$$

Theorem 3.4.1 (Kac–Rice formula). *Let $d \geqslant d_1$, where d_1 is defined by Lemma 3.2.4 and let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then for any Borel measurable function $\phi : M \to \mathbb{R}$ we have:*

$$\mathbb{E}\left[\int_{x \in Z_d} \phi(x)\,|\mathrm{d}V_d|\right] = (2\pi)^{-\frac{r}{2}} \int_{x \in M} \frac{\phi(x)}{\left|\det^{\perp}(\mathrm{ev}_x^d)\right|} \mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x) = 0\right] |\mathrm{d}V_M| \tag{3.4.1}$$

whenever one of these integrals is well-defined.

The expectation on the right-hand side of (3.4.1) is to be understood as the conditional expectation of $\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right|$ given that $s_d(x) = 0$. This result is a consequence of [Let16a, thm. 2.5.3]. See also Section 2.5.1 of [Let16a], where we applied this result with $\phi = \mathbf{1}$, in order to compute the expected volume of Z_d.

Let us denote by $\Delta = \{(x,y) \in M^2 \mid x = y\}$ the diagonal in M^2. Let $d \in \mathbb{N}$ and let $(x,y) \in M^2 \setminus \Delta$ we denote by $\mathrm{ev}_{x,y}^d$ the evaluation map:

$$\begin{array}{rccc} \mathrm{ev}_{x,y}^d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y. \\ & s & \longmapsto & (s(x), s(y)) \end{array} \tag{3.4.2}$$

The following proposition is the equivalent of Lemma 3.2.4 for two points $(x,y) \notin \Delta$. One could prove this result using only the estimates of Section 3.3. We give instead a less technical proof, using the Kodaira embedding theorem. See [MM07, sect. 5.1] for a discussion of the relations between these approaches.

Proposition 3.4.2. *There exists $d_2 \in \mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$, such that for every $d \geqslant d_2$ and every $(x,y) \in M^2 \setminus \Delta$, the evaluation map $\mathrm{ev}_{x,y}^d$ is surjective.*

Proof. Recall that there exists $d_1 \in \mathbb{N}$ such that, for all $d \geqslant d_1$, the map ev_x^d defined by (3.2.3) is surjective for any $x \in M$ (see Lem. 3.2.4). Then, for all $d \geqslant d_1$ and all $x \in M$, the complexified map $\widetilde{\mathrm{ev}}_x^d : H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ defined by $\widetilde{\mathrm{ev}}_x^d(s) = s(x)$ is also surjective.

For any $l \in \mathbb{N}$, we denote by $\Psi_l : \mathcal{X} \to \mathbb{P}\left(H^0(\mathcal{X}, \mathcal{L}^l)^*\right)$ the Kodaira map, defined by $\Psi_l(x) = \left\{s \in H^0(\mathcal{X}, \mathcal{L}^l) \mid s(x) = 0\right\}$. By the Kodaira embedding theorem (see [GH94, chap. 1.4]), there exists $l_0 \in \mathbb{N}$ such that Ψ_{l_0} is well-defined and is an embedding.

We set $d_2 = l_0 + d_1$. Let $d \geqslant d_2$ and let $(x, y) \in M^2 \setminus \Delta$. Since $\Psi_{l_0}(x)$ and $\Psi_{l_0}(y)$ are distinct hyperplanes in $H^0(\mathcal{X}, \mathcal{L}^{l_0})$, there exist σ_x and $\sigma_y \in H^0(\mathcal{X}, \mathcal{L}^{l_0})$ such that:

$$\begin{cases} \sigma_x(x) \neq 0, \\ \sigma_x(y) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \sigma_y(x) = 0, \\ \sigma_y(y) \neq 0. \end{cases}$$

Since $d - l_0 \geqslant d_1$, $\widetilde{\mathrm{ev}}_x^d$ is onto and there exist $\sigma_{1,x}, \dots, \sigma_{r,x} \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^{d-l_0})$ such that $(\sigma_{1,x}(x), \dots, \sigma_{r,x}(x))$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^{d-l_0}\right)_x$. Similarly there exist $\sigma_{1,y}, \dots, \sigma_{r,y}$ such that $(\sigma_{1,y}(y), \dots, \sigma_{r,y}(y))$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^{d-l_0}\right)_y$. We define global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$ by $s_{k,x} = \sigma_{k,x} \otimes \sigma_x$ and $s_{k,y} = \sigma_{k,y} \otimes \sigma_y$ for all $k \in \{1, \dots, r\}$. These sections are such that $(s_{k,x}(x))_{1 \leqslant k \leqslant r}$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$, $(s_{k,y}(y))_{1 \leqslant k \leqslant r}$ is a basis of $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y$ and for all $k \in \{1, \dots, r\}$, $s_{k,x}(y) = 0 = s_{k,y}(x)$. This proves that the map

$$\begin{array}{rccc} \widetilde{\mathrm{ev}}_{x,y}^d : & H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) & \longrightarrow & \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \oplus \left(\mathcal{E} \otimes \mathcal{L}^d\right)_y . \\ & s & \longmapsto & (s(x), s(y)) \end{array}$$

has rank at least $2r$ (as a $\mathbb{C}$-linear map). Since $\widetilde{\mathrm{ev}}_{x,y}^d$ is the complexified map of $\mathrm{ev}_{x,y}^d$, the latter must have rank at least $2r$ (as a $\mathbb{R}$-linear map), hence it is onto. $\square$

Remark 3.4.3. In Prop. 3.4.2, $\mathrm{ev}_{x,y}^d$ is surjective if and only if $\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right| > 0$, that is if and only if $\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*$ is non-singular. Since the latter is the variance operator of $\mathrm{ev}_{x,y}^d(s_d)$, where $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we see that the surjectivity of $\mathrm{ev}_{x,y}^d$ is equivalent to the non-degeneracy of the distribution of $(s_d(x), s_d(y))$.

We can now deduce a Kac–Rice type formula from Prop. 3.4.2. For any $d \in \mathbb{N}$, we define F_d to be the following bundle map over M^2:

$$\begin{array}{rccc} F_d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2 & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) \times \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) . \\ & (s, x, y) & \longmapsto & (s(x), s(y)) \end{array}$$

Let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ (see Def. 3.3.2). Then by Rem. 3.3.3, the restriction of ∇^d defines a connection on $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$. Let $\nabla^d F_d$ denote the vertical component of the diffential of F_d. Then, for all $(s_0, x, y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$, we have:

$$\begin{array}{rccc} \nabla_{(s_0,x,y)}^d F_d : & \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times T_x M \times T_y M & \longrightarrow & \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y . \\ & (s, v, w) & \longmapsto & \left(s(x) + \nabla_x^d s_0 \cdot v, s(y) + \nabla_y^d s_0 \cdot w\right) \end{array}$$

We denote by $\partial_1^d F_d$ the partial derivative of F_d with respect to the first variable (meaning s), and by $\partial_2^d F_d$ its partial derivative with respect to the second variable (meaning (x, y)). Then for all $(s_0, x, y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$ we have:

$$\begin{aligned} & \partial_1^d F_d(s_0, x, y) = \mathrm{ev}_{x,y}^d \\ \text{and} \qquad & \partial_2^d F_d(s_0, x, y) : (v, w) \mapsto \left(\nabla_x^d s_0 \cdot v, \nabla_y^d s_0 \cdot w\right). \end{aligned} \tag{3.4.3}$$

From now on, we assume that $d \geqslant d_2$, where d_2 is given by Prop. 3.4.2. We define an incidence manifold Σ_d by:

$$\Sigma_d = (F_d)^{-1}(0) \cap \Big(\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \big(M^2 \setminus \Delta\big)\Big).$$

By Prop. 3.4.2 and eq. 3.4.3, for all $(s, x, y) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \big(M^2 \setminus \Delta\big)$, $\partial_1^d F_{d,p}(s, x, y)$ is surjective. Thus, the restriction of F_d to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \big(M^2 \setminus \Delta\big)$ is a submersion and Σ_d is a submanifold of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$ of codimension $2r$. Note that we are only concerned with the zero set of F_d, hence none of this depends on the choice of ∇^d. We can now state the Kac–Rice formula in this context.

Theorem 3.4.4 (Kac–Rice formula)**.** *Let $d \geqslant d_2$, where d_2 is given by Proposition 3.4.2, and let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then for any Borel measurable function $\Phi : \Sigma_d \to \mathbb{R}$ we have:*

$$\mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta} \Phi(s_d, x, y)\, |\mathrm{d}V_d|^2\right] = \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta} \frac{1}{\left|\det^\perp\left(\mathrm{ev}_{x,y}^d\right)\right|}\times$$
$$\mathbb{E}\Big[\Phi(s_d, x, y)\left|\det^\perp\Big(\nabla_x^d s_d\Big)\right|\left|\det^\perp\Big(\nabla_y^d s_d\Big)\right| \,\Big|\, s_d(x) = 0 = s_d(y)\Big]\, |\mathrm{d}V_M|^2 \quad (3.4.4)$$

whenever one of these integrals is well-defined. Here, $|\mathrm{d}V_M|^2$ stands for the product measure on M^2 induced by $|\mathrm{d}V_M|$. Similarly, $|\mathrm{d}V_d|^2$ is the product measure on $(Z_d)^2$.

The expectation on the right-hand side of (3.4.4) is to be understood as the conditional expectation of $\Phi(s_d, x, y)\left|\det^\perp\left(\nabla_x^d s_d\right)\right|\left|\det^\perp\left(\nabla_y^d s_d\right)\right|$ given that $s_d(x) = 0 = s_d(y)$.

Proof. The proof of Thm. 3.4.4 uses the double fibration trick, that is apply Federer's coarea formula twice. See for example [Let16a, App. 2.C] and the reference therein.

The Euclidean inner product on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ defined by eq. (3.2.1) and the Riemannian metric g induce a Riemannian metric on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times M^2$, and on Σ_d by restriction. Let $\pi_1 : \Sigma_d \to \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and $\pi_2 : \Sigma_d \to M^2 \setminus \Delta$ denote the projections from Σ_d to the first and second factor, respectively. For all $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, $\pi_1^{-1}(s)$ is isometric to Z_s and we identify these spaces. Similarly, for all $(x, y) \in M^2 \setminus \Delta$ we identify $\pi_2^{-1}(x, y)$ with the isometric space $\ker(\mathrm{ev}_{x,y}^d)$.

We denote by $\mathrm{d}s$ the Lebesgue measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ or any of its subspaces, normalized so that a unit cube has volume 1. Let $\Phi : \Sigma_d \to \mathbb{R}$ be a Borel measurable function. Then

$$\mathbb{E}\left[\int_{(Z_d)^2\setminus\Delta} \Phi\, |\mathrm{d}V_d|^2\right] = \int_{s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)}\left(\int_{(x,y)\in\pi_1^{-1}(s)} \Phi(s, x, y)\frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\, |\mathrm{d}V_d|^2\right)\mathrm{d}s,$$

where N_d is the dimension of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, by the double fibration trick [Let16a, Prop. 2.C.3] this quantity equals:

$$\int_{(x,y)\in M^2\setminus\Delta}\left(\int_{s\in\ker(\mathrm{ev}_{x,y}^d)} \Phi(s, x, y)\frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\frac{\left|\det^\perp\left(\partial_2^d F_d(s, x, y)\right)\right|}{\left|\det^\perp\left(\partial_1^d F_d(s, x, y)\right)\right|}\,\mathrm{d}s\right)|\mathrm{d}V_M|^2. \quad (3.4.5)$$

Then eq. (3.4.3) shows that $\partial_2^d F_{d,p}(s, x, y) = \nabla_x^d s \oplus \nabla_y^d s$. Moreover, by definition of the metrics, T_xM is orthogonal to T_yM and $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$ is orthogonal to $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_y$. Thus

$$\begin{aligned}
\left|\det{}^{\perp}\left(\partial_2^d F_d(s,x,y)\right)\right| &= \det\left(\partial_2^d F_d(s,x,y)\left(\partial_2^d F_d(s,x,y)\right)^*\right)^{\frac{1}{2}}\\
&= \det\left(\begin{pmatrix}\nabla_x^d s & 0\\ 0 & \nabla_y^d s\end{pmatrix}\begin{pmatrix}(\nabla_x^d s)^* & 0\\ 0 & (\nabla_y^d s)^*\end{pmatrix}\right)^{\frac{1}{2}}\\
&= \det\begin{pmatrix}\nabla_x^d s(\nabla_x^d s)^* & 0\\ 0 & \nabla_y^d s(\nabla_y^d s)^*\end{pmatrix}^{\frac{1}{2}}\\
&= \left|\det{}^{\perp}\left(\nabla_x^d s\right)\right|\left|\det{}^{\perp}\left(\nabla_y^d s\right)\right|.
\end{aligned}$$

Besides, eq. (3.4.3) also shows that $\left|\det^{\perp}\left(\partial_1^d F_d(s,x,y)\right)\right| = \left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|$, which does not depend on s, so that (3.4.5) equals:

$$\int_{M^2\setminus\Delta}\frac{1}{\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|}\left(\int_{s\in\ker\left(\mathrm{ev}_{x,y}^d\right)}\Phi\left|\det{}^{\perp}\left(\nabla_x^d s\right)\right|\left|\det{}^{\perp}\left(\nabla_y^d s\right)\right|\frac{e^{-\frac{1}{2}\|s\|^2}}{(2\pi)^{\frac{N_d}{2}}}\,\mathrm{d}s\right)|\mathrm{d}V_M|^2 .$$

Finally, by Prop. 3.4.2, $\ker(\mathrm{ev}_x^d)$ is a subspace of codimension $2r$ of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Hence, the inner integral in (3.4.5) can be expressed as a conditional expectation given that $\mathrm{ev}_{x,y}^d(s_d)=0$, up to a factor $(2\pi)^r$. This concludes the proof of Thm. 3.4.4. □

3.4.2 An integral formula for the variance

In this section, we fix some $d \geqslant \max(d_0,d_1,d_2)$ where d_0, d_1 and d_2 are defined by Thm. 3.3.9, Lem. 3.2.4 and Prop. 3.4.2 respectively. We denote by ∇^d a real connection on $\mathcal{E}\otimes\mathcal{L}^d$. Let $\phi_1,\phi_2\in\mathcal{C}^0(M)$, we want to compute:

$$\begin{aligned}
\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1,\phi_2) &= \mathrm{Cov}(\langle|\mathrm{d}V_d|\,,\phi_1\rangle,\langle|\mathrm{d}V_d|\,,\phi_2\rangle)\\
&= \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle\langle|\mathrm{d}V_d|\,,\phi_2\rangle] - \mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_2\rangle].
\end{aligned}\tag{3.4.6}$$

First, by Thm. 3.4.1, we have:

$$\begin{aligned}
&\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_2\rangle] = \frac{1}{(2\pi)^r}\times\\
&\int_{M^2}\phi_1(x)\phi_2(y)\frac{\mathbb{E}\left[\left|\det^{\perp}(\nabla_x^d s_d)\right|\,\middle|\, s_d(x)=0\right]}{\left|\det^{\perp}(\mathrm{ev}_x^d)\right|}\frac{\mathbb{E}\left[\left|\det^{\perp}(\nabla_y^d s_d)\right|\,\middle|\, s_d(y)=0\right]}{\left|\det^{\perp}(\mathrm{ev}_y^d)\right|}|\mathrm{d}V_M|^2 .
\end{aligned}\tag{3.4.7}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}[\langle|\mathrm{d}V_d|\,,\phi_1\rangle\langle|\mathrm{d}V_d|\,,\phi_2\rangle] &= \mathbb{E}\left[\left(\int_{x\in Z_d}\phi_1(x)\,|\mathrm{d}V_d|\right)\left(\int_{y\in Z_d}\phi_2(y)\,|\mathrm{d}V_d|\right)\right]\\
&= \mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right].
\end{aligned}$$

Indeed, Z_d is almost surely of dimension $n-r>0$, so that $(Z_d)^2\cap\Delta$ (that is the diagonal in $(Z_d)^2$) has measure 0 for $|\mathrm{d}V_d|^2$. We compute this integral by Thm. 3.4.4:

$$\begin{aligned}
\mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right] &= \frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta}\frac{\phi_1(x)\phi_2(y)}{\left|\det^{\perp}\left(\mathrm{ev}_{x,y}^d\right)\right|}\times\\
&\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla_x^d s_d\right)\right|\left|\det{}^{\perp}\left(\nabla_y^d s_d\right)\right|\,\middle|\, s_d(x)=0=s_d(y)\right]|\mathrm{d}V_M|^2 .
\end{aligned}\tag{3.4.8}$$

Let $\mathcal{D}_d$ be the function defined by: $\forall (x,y) \in M^2 \setminus \Delta$,

$$\mathcal{D}_d(x,y) = \left(\frac{\mathbb{E}\left[\left|\det{}^{\perp}(\nabla^d_x s_d)\right|\left|\det{}^{\perp}(\nabla^d_y s_d)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right]}{\left|\det{}^{\perp}(\mathrm{ev}^d_{x,y})\right|} \right.$$
$$\left. - \frac{\mathbb{E}\left[\left|\det{}^{\perp}(\nabla^d_x s_d)\right| \,\middle|\, s_d(x) = 0\right] \mathbb{E}\left[\left|\det{}^{\perp}(\nabla^d_y s_d)\right| \,\middle|\, s_d(y) = 0\right]}{\left|\det{}^{\perp}(\mathrm{ev}^d_x)\right| \left|\det{}^{\perp}(\mathrm{ev}^d_y)\right|} \right) \tag{3.4.9}$$

Since $\dim M = n > 0$, Δ has measure 0 in M^2 for $|\mathrm{d}V_M|^2$. Thus, by (3.4.6), (3.4.7), (3.4.8) and (3.4.9), we have:

$$\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1, \phi_2) = \frac{1}{(2\pi)^r} \int_{M^2} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2 . \tag{3.4.10}$$

Remark 3.4.5. At this stage, it is worth noticing that the values of the conditional expectations appearing in the definition of $\mathcal{D}_d$ (see eq. (3.4.9)) do not depend on the choice of ∇^d. In fact, the whole conditional distribution of $\nabla^d_x s_d$ given that $s_d(x) = 0$ (resp. of $\nabla^d_y s_d$ given that $s_d(y) = 0$, resp. of $(\nabla^d_x s_d, \nabla^d_y s_d)$ given that $s_d(x) = 0 = s_d(y)$) is independent of the choice of ∇^d. Indeed, if $s_d(x) = 0$ then $\nabla^d_x s_d$ does not depend on ∇^d, and we conditioned on the vanishing of $s_d(x)$ (resp. $s_d(y)$, resp. $s_d(x)$ and $s_d(y)$). Thus, in the sequel, we can use any real connection we like, even one that depends on $(x,y) \in M^2 \setminus \Delta$.

3.4.3 Asymptotic for the variance

In this section we compute the asymptotic of the integral in eq. (3.4.10). The main point is to write M^2 as the disjoint union of a neighborhood of Δ, of size about $\frac{\ln d}{\sqrt{d}}$, and its complement. In (3.4.10), the set of points that are far from the diagonal will contribute a term of smaller order than the neighborhood of Δ. This is a consequence of the fast decrease of the Bergman kernel outside of the diagonal. The values of s_d at x and y are not correlated, up to some small error, outside of a neighborhood of Δ.

We still assume that $d \geqslant \max(d_0, d_1, d_2)$ and we denote by s_d a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

Asymptotics for the uncorrelated terms

Let us first compute asymptotics for the terms in the expression of $\mathcal{D}_d$ (see eq. (3.4.9)) that only depend on one point, say $x \in M$. For all $x \in M$, ev^d_x is linear. Hence $s_d(x) = \mathrm{ev}^d_x(s_d)$ is a centered Gaussian vector in $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ with variance operator:

$$\mathrm{ev}^d_x \left(\mathrm{ev}^d_x\right)^* = \mathbb{E}[s_d(x) \otimes (s_d(x))^*] = E_d(x,x), \tag{3.4.11}$$

where E_d is the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$ and the last equality is given by Prop. 3.2.6.

Lemma 3.4.6. *For every $x \in M$, we have:*

$$\left(\frac{\pi}{d}\right)^{\frac{nr}{2}} \left|\det{}^{\perp}\left(\mathrm{ev}^d_x\right)\right| = 1 + O\left(d^{-1}\right),$$

where the error term $O\left(d^{-1}\right)$ does not depend on x.

Proof. Let $x \in M$, then $\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|^2 = \det E_d(x,x)$ by (3.4.11). By (3.3.8), we have:

$$\left(\frac{\pi}{d}\right)^{nr}\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|^2 = \det\left(\mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} + O\left(d^{-1}\right)\right) = 1 + O\left(d^{-1}\right).$$

The error term in (3.3.8) is independent of x, therefore the same is true here. □

Let ∇^d be a real connection on $\mathcal{E}\otimes\mathcal{L}^d$. We assume that ∇^d is a metric connection, so that Lem. 3.2.10 and Cor. 3.2.12 are valid in this context. Recall that the Chern connection is an example of real metric connection.

For all $x \in M$, let $j_x^d : s \mapsto \left(s(x), \nabla_x^d s\right)$ denote the evaluation of the 1-jet at x, from $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ to $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes (\mathbb{R}\oplus T_x^*M)$. Since j_x^d is linear, $\left(s_d(x), \nabla_x^d s_d\right)$ is a centered Gaussian vector with variance operator $j_x^d\left(j_x^d\right)^*$. This operator splits according to the direct sum $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_x^*M$:

$$\begin{aligned} j_x^d\left(j_x^d\right)^* &= \mathbb{E}\left[j_x^d(s_d)\otimes\left(j_x^d(s_d)\right)^*\right] \\ &= \begin{pmatrix} \mathbb{E}\left[s_d(x)\otimes s_d(x)^*\right] & \mathbb{E}\left[s_d(x)\otimes\left(\nabla_x^d s_d\right)^*\right] \\ \mathbb{E}\left[\nabla_x^d s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla_x^d s_d\otimes\left(\nabla_x^d s_d\right)^*\right]\end{pmatrix} \\ &= \begin{pmatrix} E_d(x,x) & \partial_y^\sharp E_d(x,x) \\ \partial_x E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,x)\end{pmatrix}, \end{aligned} \tag{3.4.12}$$

where the last equality comes from Cor. 3.2.13. We chose $d \geqslant d_1$, so that ev_x^d is surjective (see Lem. 3.2.4), i.e. $\det\left(\mathrm{ev}_x^d\left(\mathrm{ev}_x^d\right)^*\right) > 0$. Hence, the distribution of $s_d(x)$ is non-degenerate. Then (see [AW09, prop. 1.2]), the distribution of $\nabla_x^d s_d$ given that $s_d(x) = 0$ is a centered Gaussian in $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_x^*M$ with variance operator:

$$\partial_x\partial_y^\sharp E_d(x,x) - \partial_x E_d(x,x)\left(E_d(x,x)\right)^{-1}\partial_y^\sharp E_d(x,x). \tag{3.4.13}$$

Lemma 3.4.7. *For every $x \in M$, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{r}{2}}\mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x) = 0\right] = (2\pi)^{\frac{r}{2}}\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\left(1 + O\left(d^{-1}\right)\right),$$

where the error term is independent of x.

Proof. Let $x \in M$, and let $L_d(x)$ be a centered Gaussian vector in $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_x^*M$ with variance operator:

$$\Lambda_d(x) = \frac{\pi^n}{d^{n+1}}\left(\partial_x\partial_y^\sharp E_d(x,x) - \partial_x E_d(x,x)\left(E_d(x,x)\right)^{-1}\partial_y^\sharp E_d(x,x)\right). \tag{3.4.14}$$

By (3.4.13) and the above discussion, the distribution of $\nabla_x^d s_d$ given that $s_d(x) = 0$ equals that of $\left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{1}{2}} L_d(x)$. Then,

$$\begin{aligned} \mathbb{E}\left[\left|\det^{\perp}\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x) = 0\right] &= \mathbb{E}\left[\left|\det^{\perp}\left(\left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{1}{2}} L_d(x)\right)\right|\right] \\ &= \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}}\mathbb{E}\left[\left|\det^{\perp}\left(L_d(x)\right)\right|\right]. \end{aligned} \tag{3.4.15}$$

Recall that the distribution of $\nabla_x^d s_d$ given that $s_d(x) = 0$ does not depend on the choice of ∇^d (Rem. 3.4.5). Hence $\Lambda_d(x)$ does not depend on the choice of ∇^d. For the following computation, we choose ∇^d to be trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Then we can use the diagonal estimates of Cor. 3.3.8 for the Bergman kernel and its derivatives. We have: $\Lambda_d(x) = \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \otimes \mathrm{Id}_{T_x^*M} + O(d^{-1})$, where the error does not depend on x. Hence,

$$\det(\Lambda_d(x)) = 1 + O\big(d^{-1}\big)\,. \tag{3.4.16}$$

Besides, there exists some $K > 0$ such that $\left\|\Lambda_d(x)^{-1} - \mathrm{Id}\right\| \leqslant Kd^{-1}$ for all d large enough. Then, by the mean value inequality, for all $L \in \mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_x^*M$

$$\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x)^{-1} - \mathrm{Id}\right)L\,,L\right\rangle\right) - 1\right| \leqslant \frac{K}{2d}\left\|L\right\|^2\exp\left(\frac{K}{2d}\left\|L\right\|^2\right).$$

Let $L_d^0(x) \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_x^*M$ and let $\mathrm{d}L$ denote the normalized Lebesgue measure on $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x \otimes T_x^*M$. Then we have:

$$\begin{aligned}
(2\pi)^{\frac{nr}{2}}&\left|\det(\Lambda_d(x))^{\frac{1}{2}}\,\mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right|\right] - \mathbb{E}\left[\left|\det{}^{\perp}\big(L_d^0(x)\big)\right|\right]\right| \\
&\leqslant \int\left|\det{}^{\perp}(L)\right|e^{-\frac{1}{2}\|L\|^2}\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x)^{-1} - \mathrm{Id}\right)L\,,L\right\rangle\right) - 1\right|\mathrm{d}L \\
&\leqslant \frac{K}{2d}\int\left|\det{}^{\perp}(L)\right|\exp\left(-\frac{1}{2}\left(1 - \frac{K}{d}\right)\|L\|^2\right)\mathrm{d}L.
\end{aligned}$$

The integral on the last line converges to some finite limit as $d \to +\infty$. Thus, by (3.4.16),

$$\begin{aligned}
\mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right|\right] &= \det(\Lambda_d(x))^{-\frac{1}{2}}\left(\mathbb{E}\left[\left|\det{}^{\perp}\big(L_d^0(x)\big)\right|\right] + O\big(d^{-1}\big)\right) \\
&= \mathbb{E}\left[\left|\det{}^{\perp}\big(L_d^0(x)\big)\right|\right] + O\big(d^{-1}\big)\,,
\end{aligned} \tag{3.4.17}$$

uniformly in $x \in M$. Lemma 3.4.7 follows from (3.4.15), (3.4.17) and the following equality, that was proved in [Let16a, lem. 2.A.14]:

$$\mathbb{E}\left[\left|\det{}^{\perp}\big(L_d^0(x)\big)\right|\right] = (2\pi)^{\frac{r}{2}}\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}. \tag{3.4.18}$$

□

Far off-diagonal asymptotics for the correlated terms

We can now focus on computing terms in the expression of $\mathcal{D}_d$ that depend on both x and y. For all $(x,y) \in M^2 \setminus \Delta$, $\mathrm{ev}_{x,y}^d(s_d) = (s_d(x), s_d(y))$ is a centered Gaussian vector with variance operator:

$$\begin{aligned}
\mathrm{ev}_{x,y}^d(\mathrm{ev}_{x,y}^d)^* &= \mathbb{E}\left[\mathrm{ev}_{x,y}^d(s_d)\otimes\mathrm{ev}_{x,y}^d(s_d)^*\right] \\
&= \begin{pmatrix}\mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}[s_d(x)\otimes s_d(y)^*] \\ \mathbb{E}[s_d(y)\otimes s_d(x)^*] & \mathbb{E}[s_d(y)\otimes s_d(y)^*]\end{pmatrix} \\
&= \begin{pmatrix}E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix},
\end{aligned} \tag{3.4.19}$$

where we decomposed this operator according to the direct sum $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y$. Since we assumed $d \geqslant d_2$, $\left|\det{}^{\perp}\big(\mathrm{ev}_{x,y}^d\big)\right| > 0$ (see Prop. 3.4.2) and the distribution of $(s_d(x), s_d(y))$ is non-degenerate.

We denote by $j_{x,y}^d : s \mapsto \big(s(x), s(y), \nabla_x^d s, \nabla_y^d s\big)$ the evaluation of the 1-jet at (x,y). Then $j_{x,y}^d$ is a linear map from $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ to

$$\mathbb{R}\Big(\mathcal{E}\otimes\mathcal{L}^d\Big)_x \oplus \mathbb{R}\Big(\mathcal{E}\otimes\mathcal{L}^d\Big)_y \oplus \left(\mathbb{R}\Big(\mathcal{E}\otimes\mathcal{L}^d\Big)_x \otimes T_x^*M\right) \oplus \left(\mathbb{R}\Big(\mathcal{E}\otimes\mathcal{L}^d\Big)_y \otimes T_y^*M\right), \tag{3.4.20}$$

and $j^d_{x,y}(s_d)$ is a centered Gaussian vector, with variance operator $j^d_{x,y}\left(j^d_{x,y}\right)^*$. We can split this variance operator according to the direct sum (3.4.20). Then by Cor. 3.2.13, we have:

$$
\begin{aligned}
&j^d_{x,y}\left(j^d_{x,y}\right)^* = \mathbb{E}\left[j^d_{x,y}(s_d)\otimes\left(j^d_{x,y}(s_d)\right)^*\right]\\
&= \begin{pmatrix}
\mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}[s_d(x)\otimes s_d(y)^*] & \mathbb{E}\left[s_d(x)\otimes (\nabla^d_x s_d)^*\right] & \mathbb{E}\left[s_d(x)\otimes (\nabla^d_y s_d)^*\right]\\
\mathbb{E}[s_d(y)\otimes s_d(x)^*] & \mathbb{E}[s_d(y)\otimes s_d(y)^*] & \mathbb{E}\left[s_d(y)\otimes (\nabla^d_x s_d)^*\right] & \mathbb{E}\left[s_d(y)\otimes (\nabla^d_y s_d)^*\right]\\
\mathbb{E}\left[\nabla^d_x s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla^d_x s_d\otimes s_d(y)^*\right] & \mathbb{E}\left[\nabla^d_x s_d\otimes (\nabla^d_x s_d)^*\right] & \mathbb{E}\left[\nabla^d_x s_d\otimes (\nabla^d_y s_d)^*\right]\\
\mathbb{E}\left[\nabla^d_y s_d\otimes s_d(x)^*\right] & \mathbb{E}\left[\nabla^d_y s_d\otimes s_d(y)^*\right] & \mathbb{E}\left[\nabla^d_y s_d\otimes (\nabla^d_x s_d)^*\right] & \mathbb{E}\left[\nabla^d_y s_d\otimes (\nabla^d_y s_d)^*\right]
\end{pmatrix}\\
&= \begin{pmatrix}
E_d(x,x) & E_d(x,y) & \partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y)\\
E_d(y,x) & E_d(y,y) & \partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y)\\
\partial_x E_d(x,x) & \partial_x E_d(x,y) & \partial_x\partial^\sharp_y E_d(x,x) & \partial_x\partial^\sharp_y E_d(x,y)\\
\partial_x E_d(y,x) & \partial_x E_d(y,y) & \partial_x\partial^\sharp_y E_d(y,x) & \partial_x\partial^\sharp_y E_d(y,y)
\end{pmatrix}.
\end{aligned}
\tag{3.4.21}
$$

Since the distribution of $(s_d(x), s_d(y))$ is non-degenerate, the distribution of $\left(\nabla^d_x s, \nabla^d_y s\right)$ given that $\mathrm{ev}_{x,y}(s_d) = 0$ is a centered Gaussian with variance operator:

$$
\begin{aligned}
&\begin{pmatrix}
\partial_x\partial^\sharp_y E_d(x,x) & \partial_x\partial^\sharp_y E_d(x,y)\\
\partial_x\partial^\sharp_y E_d(y,x) & \partial_x\partial^\sharp_y E_d(y,y)
\end{pmatrix} -\\
&\qquad\begin{pmatrix}
\partial_x E_d(x,x) & \partial_x E_d(x,y)\\
\partial_x E_d(y,x) & \partial_x E_d(y,y)
\end{pmatrix}
\begin{pmatrix}
E_d(x,x) & E_d(x,y)\\
E_d(y,x) & E_d(y,y)
\end{pmatrix}^{-1}
\begin{pmatrix}
\partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y)\\
\partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y)
\end{pmatrix}.
\end{aligned}
\tag{3.4.22}
$$

Definition 3.4.8. For every $(x,y) \in M^2\backslash\Delta$ and every d large enough, we define $\Lambda_d(x,y)$ to be the operator such that $\frac{d^{n+1}}{\pi^n}\Lambda_d(x,y)$ equals (3.4.22). That is, $\Lambda_d(x,y)$ is the conditional variance of $\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}}\left(\nabla^d_x s, \nabla^d_y s\right)$ given that $\mathrm{ev}_{x,y}(s_d) = 0$.

Let $C' > 0$ be the constant appearing in the exponential in Thm. 3.3.9. We denote

$$
b_n = \frac{1}{C'}\left(\frac{n}{2}+1\right)
\tag{3.4.23}
$$

and

$$
\Delta_d = \left\{(x,y)\in M^2 \;\middle|\; \rho_g(x,y) < b_n\frac{\ln d}{\sqrt{d}}\right\},
\tag{3.4.24}
$$

where, as before, ρ_g is the geodesic distance in (M,g).

Lemma 3.4.9. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$
\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right| = \left|\det{}^\perp\left(\mathrm{ev}^d_x\right)\right|\left|\det{}^\perp\left(\mathrm{ev}^d_y\right)\right|\left(1+O\left(d^{-\frac{n}{2}-1}\right)\right),
$$

where the error term is uniform in $(x,y) \in M^2 \setminus \Delta_d$

Proof. For all $(x,y) \in M^2 \setminus \Delta_d$, we have $\rho_g(x,y) \geqslant b_n\frac{\ln d}{\sqrt{d}}$. Then, by Thm. 3.3.9,

$$
\|E_d(x,y)\| \leqslant C_0 d^n \exp\left(-C' b_n \ln d\right) \leqslant C_0 d^{\frac{n}{2}-1}.
$$

Then, by (3.4.19) we have:

$$
\begin{aligned}
\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^* &= \begin{pmatrix} E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}\\
&= \begin{pmatrix} E_d(x,x) & 0\\ 0 & E_d(y,y)\end{pmatrix} + O\left(d^{\frac{n}{2}-1}\right).
\end{aligned}
$$

Besides, by (3.3.8),

$$\begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix}^{-1} = \left(\frac{\pi}{d}\right)^n \left(\mathrm{Id} + O\big(d^{-1}\big)\right) = O\big(d^{-n}\big), \tag{3.4.25}$$

so that

$$\begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix} = \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} \left(\mathrm{Id} + O\Big(d^{-\frac{n}{2}-1}\Big)\right). \tag{3.4.26}$$

We conclude the proof by taking the square root of the determinant of this last equality (recall (3.4.11)). $\square$

Lemma 3.4.10. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$\begin{aligned} &\mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right] = \\ &\qquad \mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right| \,\middle|\, s_d(x) = 0\right] \mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right| \,\middle|\, s_d(y) = 0\right] \left(1 + O\Big(d^{-\frac{n}{2}-1}\Big)\right), \end{aligned}$$

where the error term is uniform in $(x,y) \in M^2 \setminus \Delta_d$

This lemma is a consequence of the following technical result.

Lemma 3.4.11. *For every $(x,y) \in M^2 \setminus \Delta_d$, we have:*

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix} \left(\mathrm{Id} + O\Big(d^{-\frac{n}{2}-1}\Big)\right),$$

uniformly in $(x,y) \in M^2 \setminus \Delta_d$.

Proof of Lemma 3.4.10. Let $(L_d(x), L_d(y))$ and $(L'_d(x), L'_d(y))$ be centered Gaussian vectors in

$$\left(\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T^*_x M\right) \oplus \left(\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_y \otimes T^*_y M\right)$$

such that: the variance of $(L'_d(x), L'_d(y))$ is $\Lambda_d(x,y)$ (recall Def. 3.4.8), and $L_d(x)$ and $L_d(y)$ are independent with variances $\Lambda_d(x)$ and $\Lambda_d(y)$ respectively (see (3.4.14)). Then, the distribution of $(L_d(x), L_d(y))$ is a centered Gaussian with variance $\left(\begin{smallmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{smallmatrix}\right)$. From the definitions of $\Lambda_d(x)$, $\Lambda_d(y)$ and $\Lambda_d(x,y)$, we have:

$$\begin{aligned} &\mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right| \,\middle|\, s_d(x) = 0\right] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}} \mathbb{E}\left[\left|\det{}^\perp(L_d(x))\right|\right], \\ &\mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right| \,\middle|\, s_d(y) = 0\right] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}} \mathbb{E}\left[\left|\det{}^\perp(L_d(y))\right|\right], \\ &\mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right] = \\ &\qquad\qquad \left(\frac{d^{n+1}}{\pi^n}\right)^{r} \mathbb{E}\left[\left|\det{}^\perp(L'_d(x))\right|\left|\det{}^\perp(L'_d(y))\right|\right]. \end{aligned}$$

Since $L_d(x)$ and $L_d(y)$ are independent, we only need to prove that:

$$\mathbb{E}\left[\left|\det{}^\perp(L'_d(x))\right|\left|\det{}^\perp(L'_d(y))\right|\right] = \mathbb{E}\left[\left|\det{}^\perp(L_d(x))\right|\left|\det{}^\perp(L_d(y))\right|\right]\left(1 + O\Big(d^{-\frac{n}{2}-1}\Big)\right). \tag{3.4.27}$$

By Lemma 3.4.11,

$$\begin{aligned}\det\left(\Lambda_d(x,y)\right) &= \det\left(\begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}\left(\mathrm{Id}+O\left(d^{-\frac{n}{2}-1}\right)\right)\right)\\ &= \det\left(\Lambda_d(x)\right)\det\left(\Lambda_d(x)\right)\left(1+O\left(d^{-\frac{n}{2}-1}\right)\right).\end{aligned} \tag{3.4.28}$$

Besides Lem. 3.4.11 shows that:

$$\Lambda_d(x,y)^{-1} = \begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}^{-1}\left(\mathrm{Id}+O\left(d^{-\frac{n}{2}-1}\right)\right).$$

By Cor. 3.3.8 and eq. (3.4.14), we have: $\left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right) = \mathrm{Id}+O\left(d^{-1}\right)$. Hence,

$$\begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}^{-1} = \mathrm{Id}+O\left(d^{-1}\right) \tag{3.4.29}$$

uniformly in (x,y), and

$$\Lambda_d(x,y)^{-1} - \begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}^{-1} = O\left(d^{-\frac{n}{2}-1}\right).$$

Thus there exists $K>0$ such that, for all d large enough,

$$\left\|\Lambda_d(x,y)^{-1} - \begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}^{-1}\right\| \leqslant \frac{K}{d^{\frac{n}{2}+1}}.$$

Then, for every $L=(L_1,L_2)\in\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes T_x^*M\right)\oplus\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y\otimes T_y^*M\right)$, we have:

$$\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x,y)^{-1} - \left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right)^{-1}\right)L\,,L\right\rangle\right)-1\right| \leqslant \frac{K\left\|L\right\|^2}{2d^{\frac{n}{2}+1}}\exp\left(\frac{K\left\|L\right\|^2}{2d^{\frac{n}{2}+1}}\right).$$

Let $\mathrm{d}L$ denote the normalized Lebesgue measure on this vector space. We have:

$$\begin{aligned}&(2\pi)^{nr}\left|\det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}}\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right|\left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right]\right.\\ &\qquad\qquad\left.-\det\left(\Lambda_d(x)\right)^{\frac{1}{2}}\det\left(\Lambda_d(y)\right)^{\frac{1}{2}}\mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right|\left|\det{}^{\perp}(L_d(y))\right|\right]\right|\\ &\leqslant \int\left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\exp\left(-\frac{1}{2}\left\langle\left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right)^{-1}L\,,L\right\rangle\right)\times\\ &\qquad\qquad\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x,y)^{-1} - \left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right)^{-1}\right)L\,,L\right\rangle\right)-1\right|\mathrm{d}L\\ &\leqslant \frac{K}{2d^{\frac{n}{2}+1}}\int\left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\left\|L\right\|^2\times\\ &\qquad\qquad\exp\left(-\frac{1}{2}\left\langle\left(\left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right)^{-1}-\frac{K}{2d^{\frac{n}{2}+1}}\mathrm{Id}\right)L\,,L\right\rangle\right)\mathrm{d}L\\ &= O\left(d^{-\frac{n}{2}-1}\right).\end{aligned}$$

Let us prove the last equality. By eq. (3.4.29), for every d large enough (uniform in (x,y)),

$$\left\|\begin{pmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{pmatrix}^{-1} - \left(1+\frac{K}{2d^{\frac{n}{2}+1}}\right)\mathrm{Id}\right\| \leqslant \frac{1}{2},$$

so that:

$$\int \left|\det{}^{\perp}(L_1)\right| \left|\det{}^{\perp}(L_2)\right| \|L\|^2 \exp\left(-\frac{1}{2}\left\langle\left(\left(\begin{smallmatrix}\Lambda_d(x) & 0\\ 0 & \Lambda_d(y)\end{smallmatrix}\right)^{-1} - \frac{K}{2d^{\frac{n}{2}+1}}\operatorname{Id}\right) L\,, L\right\rangle\right) \mathrm{d}L$$
$$\leqslant \int \left|\det{}^{\perp}(L_1)\right| \left|\det{}^{\perp}(L_2)\right| \|L\|^2 \exp\left(-\frac{1}{4}\|L\|^2\right) \mathrm{d}L.$$

And the last integral is finite since $\left|\det^{\perp}(L_1)\right| \left|\det^{\perp}(L_2)\right| \|L\|^2$ is the norm of a polynomial in L.

Eq. (3.4.16) and (3.4.28) show that $\det(\Lambda_d(x,y)) = 1 + O\big(d^{-1}\big)$. Then, by the previous computations and eq. (3.4.28), we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right| \left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right]\\
&\qquad= \left(\frac{\det\left(\Lambda_d(x)\right)\det\left(\Lambda_d(y)\right)}{\det\left(\Lambda_d(x,y)\right)}\right)^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right| \left|\det{}^{\perp}(L_d(y))\right|\right] + O\left(d^{-\frac{n}{2}-1}\right)\\
&\qquad= \mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right| \left|\det{}^{\perp}(L_d(y))\right|\right]\left(1 + O\left(d^{-\frac{n}{2}-1}\right)\right) + O\left(d^{-\frac{n}{2}-1}\right).
\end{aligned}$$

Equations (3.4.17) and (3.4.18) proves that

$$\mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right| \left|\det{}^{\perp}(L_d(y))\right|\right] = \mathbb{E}\left[\left|\det{}^{\perp}(L_d(x))\right|\right] \mathbb{E}\left[\left|\det{}^{\perp}(L_d(y))\right|\right]$$

converges to some positive constant. This proves (3.4.27) and establishes Lemma 3.4.10. □

Proof of Lemma 3.4.11. First, recall that $\Lambda_d(x,y)$, $\Lambda_d(x)$ and $\Lambda_d(y)$ do not depend on the choice of ∇^d (see Rem. 3.4.5). In this proof, we use the Chern connection which is both real and metric. Let $(x,y) \in M^2 \setminus \Delta_d$, then $\rho_g(x,y) \geqslant b_n \frac{\ln d}{\sqrt{d}}$. By Thm. 3.3.9, we have:

$$\|\partial_x E_d(x,y)\| \leqslant C_1 d^{n+\frac{1}{2}} \exp\left(-C' b_n \ln d\right) \leqslant C_1 d^{\frac{n}{2}-\frac{1}{2}}.$$

Similarly, $\|\partial_x E_d(y,x)\|$, $\left\|\partial_y^{\sharp} E_d(x,y)\right\|$ and $\left\|\partial_y^{\sharp} E_d(y,x)\right\|$ are smaller than $C_1 d^{\frac{n-1}{2}}$. Then

$$\begin{pmatrix}\partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix} = \begin{pmatrix}\partial_x E_d(x,x) & 0\\ 0 & \partial_x E_d(y,y)\end{pmatrix} + O\left(d^{\frac{n-1}{2}}\right) \tag{3.4.30}$$

$$\begin{pmatrix}\partial_y^{\sharp} E_d(x,x) & \partial_y^{\sharp} E_d(x,y)\\ \partial_y^{\sharp} E_d(y,x) & \partial_y^{\sharp} E_d(y,y)\end{pmatrix} = \begin{pmatrix}\partial_y^{\sharp} E_d(x,x) & 0\\ 0 & \partial_y^{\sharp} E_d(y,y)\end{pmatrix} + O\left(d^{\frac{n-1}{2}}\right) \tag{3.4.31}$$

and, by eq. (3.4.26),

$$\begin{pmatrix}E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}^{-1} = \begin{pmatrix}E_d(x,x) & 0\\ 0 & E_d(y,y)\end{pmatrix}^{-1}\left(\operatorname{Id} + O\left(d^{-\frac{n}{2}-1}\right)\right). \tag{3.4.32}$$

Using eq. (3.3.9), (3.3.10) and (3.4.25), we get:

$$\begin{aligned}
&\begin{pmatrix}\partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix}\begin{pmatrix}E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}^{-1}\begin{pmatrix}\partial_y^{\sharp} E_d(x,x) & \partial_y^{\sharp} E_d(x,y)\\ \partial_y^{\sharp} E_d(y,x) & \partial_y^{\sharp} E_d(y,y)\end{pmatrix}\\
&= \begin{pmatrix}\partial_x E_d(x,x) & 0\\ 0 & \partial_x E_d(y,y)\end{pmatrix}\begin{pmatrix}E_d(x,x) & 0\\ 0 & E_d(y,y)\end{pmatrix}^{-1}\begin{pmatrix}\partial_y^{\sharp} E_d(x,x) & 0\\ 0 & \partial_y^{\sharp} E_d(y,y)\end{pmatrix}\\
&\qquad + O\left(d^{\frac{n}{2}-1}\right)
\end{aligned} \tag{3.4.33}$$

Using Thm. 3.3.9 once more, we know that $\left\|\partial_x\partial_y^\sharp E_d(x,y)\right\|$ and $\left\|\partial_x\partial_y^\sharp E_d(y,x)\right\|$ are smaller than $C_2 d^{\frac{n}{2}}$. Then we have:

$$\begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y) \\ \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix} = \begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & 0 \\ 0 & \partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix} + O\left(d^{\frac{n}{2}}\right). \tag{3.4.34}$$

We substract eq. (3.4.33) to eq. (3.4.34) and divide by $\frac{d^{n+1}}{\pi^n}$. By definition of $\Lambda_d(x,y)$, $\Lambda_d(x)$ and $\Lambda_d(y)$ (see Def. 3.4.8 and eq. (3.4.14)),

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix} + O\left(d^{-\frac{n}{2}-1}\right) = \begin{pmatrix} \Lambda_d(x) & 0 \\ 0 & \Lambda_d(y) \end{pmatrix} \left(\mathrm{Id} + O\left(d^{-\frac{n}{2}-1}\right)\right),$$

where we used the fact that $\Lambda_d(x) = \mathrm{Id} + O(d^{-1}) = \Lambda_d(y)$ to obtain the last equality. $\square$

Proposition 3.4.12. *Let $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, then we have the following as $d \to +\infty$:*

$$\int_{M^2\setminus\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2 = \|\phi_1\|_\infty \|\phi_2\|_\infty O\left(d^{r-\frac{n}{2}-1}\right),$$

where the error term is independent of (ϕ_1, ϕ_2).

Proof. We combine Lemmas 3.4.9 and 3.4.10, which gives:

$$\begin{aligned} &\frac{\mathbb{E}\left[\left|\det^\perp\left(\nabla_x^d s_d\right)\right| \left|\det^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d) = 0\right]}{\left|\det^\perp\left(\mathrm{ev}_{x,y}^d\right)\right|} = \\ &\qquad \frac{\mathbb{E}\left[\left|\det^\perp\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x) = 0\right] \mathbb{E}\left[\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, s_d(y) = 0\right]}{\left|\det^\perp(\mathrm{ev}_x^d)\right| \left|\det^\perp\left(\mathrm{ev}_y^d\right)\right|} \left(1 + O\left(d^{-\frac{n}{2}-1}\right)\right). \end{aligned}$$

for all $(x,y) \in M^2 \setminus \Delta_d$. Besides, by Lemmas 3.4.6 and 3.4.7,

$$\frac{\mathbb{E}\left[\left|\det^\perp\left(\nabla_x^d s_d\right)\right| \,\middle|\, s_d(x) = 0\right]}{\left|\det^\perp(\mathrm{ev}_x^d)\right|} = O\left(d^{\frac{r}{2}}\right) = \frac{\mathbb{E}\left[\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, s_d(y) = 0\right]}{\left|\det^\perp\left(\mathrm{ev}_y^d\right)\right|}.$$

Recalling the definition of $\mathcal{D}_d$ (eq. (3.4.9)), we obtain that:

$$\forall (x,y) \in M^2 \setminus \Delta_d, \qquad \mathcal{D}_d(x,y) = O\left(d^{r-\frac{n}{2}-1}\right),$$

uniformly in $(x,y) \notin \Delta_d$. Then, for any continuous ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, we have:

$$\begin{aligned} \left|\int_{M^2\setminus\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2\right| &\leqslant \|\phi_1\|_\infty \|\phi_2\|_\infty \mathrm{Vol}\left(M^2\right) \left(\sup_{M^2\setminus\Delta_d} |\mathcal{D}_d|\right) \\ &= \|\phi_1\|_\infty \|\phi_2\|_\infty O\left(d^{r-\frac{n}{2}-1}\right). \end{aligned}$$

and the error term does not depend on (ϕ_1, ϕ_2). $\square$

Properties of the limit distribution

Before we tackle the computation of the dominant term in (3.4.10), that is the integral over Δ_d, we introduce the random variables that will turn out to be the scaling limits of $\left(\nabla^d_x s_d, \nabla^d_y s_d\right)$ given that $\mathrm{ev}^d_{x,y}(s_d) = 0$. We also establish some of their properties.

Notation 3.4.13. Let $x \in M$ and $z \in T_xM$, we denote by $z^* \otimes z \in T^*_xM \otimes T_xM$ the linear map:

$$\begin{array}{rccc} z^* \otimes z : & T^*_xM & \longrightarrow & T^*_xM. \\ & \eta & \longmapsto & \eta(z)z^* \end{array}$$

Let $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ be an orthonormal basis of T_xM and let $(dx_1, \dots, dx_n)$ denote its dual basis. If $z = \sum z_i \frac{\partial}{\partial x_i}$ then $z^* \otimes z = \sum z_i z_j dx_i \otimes \frac{\partial}{\partial x_j}$, i.e. the matrix of $z^* \otimes z$ in $(dx_1, \dots, dx_n)$ is $(z_i z_j)_{1 \leqslant i,j \leqslant n}$.

Definition 3.4.14. For all $x \in M$ and $z \in T_xM \setminus \{0\}$, we define

$$\Lambda_x(z) \in \mathrm{End}\left(\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T^*_xM \otimes \mathbb{R}^2\right)$$

by:

$$\Lambda_x(z) = \begin{pmatrix} \mathrm{Id}_{T^*_xM} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T^*_xM} - \frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T^*_xM} - \frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right) & \mathrm{Id}_{T^*_xM} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x}.$$

We need information about $\Lambda_x(z)$, especially concerning the vanishing of its eigenvalues. This will be useful in the estimates involving $\Lambda_x(z)$ below.

Lemma 3.4.15. *For all $x \in M$ and $z \in T_xM \setminus \{0\}$, the eigenvalues of $\Lambda_x(z)$ are:*

- $1 - e^{-\frac{1}{2}\|z\|^2}$ *and* $1 + e^{-\frac{1}{2}\|z\|^2}$*, each with multiplicity $(n-1)r$,*
- $\dfrac{1 - e^{-\|z\|^2} + \|z\|^2 e^{-\frac{1}{2}\|z\|^2}}{1 + e^{-\frac{1}{2}\|z\|^2}}$ *and* $\dfrac{1 - e^{-\|z\|^2} - \|z\|^2 e^{-\frac{1}{2}\|z\|^2}}{1 - e^{-\frac{1}{2}\|z\|^2}}$*, each with multiplicity r.*

Proof. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. By definition of $\Lambda_x(z)$, its eigenvalues are the same as that of

$$\begin{pmatrix} \mathrm{Id}_{T^*_xM} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T^*_xM} - \frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T^*_xM} - \frac{z^*\otimes z}{1-e^{-\|z\|^2}}\right) & \mathrm{Id}_{T^*_xM} - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix}, \tag{3.4.35}$$

with multiplicities multiplied by r. Hence, it is enough to compute the eigenvalues of the operator (3.4.35).

Let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ of T_xM such that $z = \|z\| \frac{\partial}{\partial x_1}$, and let us denote by $(dx_1, \dots, dx_n)$ the dual basis. Then, $z^* \otimes z = \|z\|^2 dx_1 \otimes \frac{\partial}{\partial x_1}$. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$, then the matrix of the operator (3.4.35) in the orthonormal basis $(e_1 \otimes dx_1, e_2 \otimes dx_1, \dots, e_1 \otimes dx_n, \dots, e_2 \otimes dx_n)$ is:

$$\left(\begin{array}{cc|c} 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0 \\ e^{-\frac{1}{2}\|z\|^2}\left(1 - \frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 1 - \frac{\|z\|^2 e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 \\ \hline 0 & 0 & \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right), \tag{3.4.36}$$

where I_{n-1} stands for the identity matrix of size $n-1$.

The bottom-right block has eigenvalues $1 - e^{-\frac{1}{2}\|z\|^2}$ and $1 + e^{-\frac{1}{2}\|z\|^2}$, each with multiplicity $n-1$. To conclude the proof of Lemma 3.4.15, we only need to observe that, for all $t > 0$, the eigenvalues of

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) \\ e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) & 1 - \frac{te^{-t}}{1-e^{-t}} \end{pmatrix}$$

are:

$$1 - \frac{te^{-t}}{1-e^{-t}} + e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) = \frac{1 - e^{-t} - te^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}}$$

and

$$1 - \frac{te^{-t}}{1-e^{-t}} - e^{-\frac{1}{2}t}\left(1 - \frac{t}{1-e^{-t}}\right) = \frac{1 - e^{-t} + te^{-\frac{1}{2}t}}{1 + e^{-\frac{1}{2}t}}.$$

Note that the latter one is the largest. □

Definition 3.4.16. We define the function $f : (0, +\infty) \to \mathbb{R}$ by:

$$\forall t > 0, \qquad f(t) = \frac{1 - e^{-\frac{1}{2}t}}{1 - e^{-t} - te^{-\frac{1}{2}t}}.$$

Corollary 3.4.17. *Let $x \in M$ and $z \in T_xM \setminus \{0\}$, then we have:*

$$\det(\Lambda_x(z)) = \\ \left(1 - e^{-\|z\|^2}\right)^{r(n-2)} \left(1 - e^{-\|z\|^2} + \|z\|^2 e^{-\frac{1}{2}\|z\|^2}\right)^r \left(1 - e^{-\|z\|^2} - \|z\|^2 e^{-\frac{1}{2}\|z\|^2}\right)^r > 0. \tag{3.4.37}$$

Moreover,

$$\|\Lambda_x(z)\| < 2 \qquad \text{and} \qquad \left\|\Lambda_x(z)^{-1}\right\| = f\left(\|z\|^2\right), \tag{3.4.38}$$

where $\|\cdot\|$ denote the operator norm on $\mathrm{End}\left(\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T_x^*M\right)$.

Proof. First, the formula for $\det(\Lambda_x(z))$ is a direct consequence of Lem. 3.4.15, and we only need to check that the eigenvalues of $\Lambda_x(z)$ are positive. Clearly, $1 \pm e^{-\frac{1}{2}t} > 0$ when $t > 0$.

Then, for all positive t, we have:

$$\frac{1 - e^{-t} - te^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}} = \frac{e^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} - t\right) = \frac{e^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}}\left(2\sinh\left(\frac{t}{2}\right) - t\right),$$

and $2\sinh\left(\frac{t}{2}\right) > t$. Besides,

$$\frac{1 - e^{-t} + te^{-\frac{1}{2}t}}{1 + e^{-\frac{1}{2}t}} = \frac{e^{-\frac{1}{2}t}}{1 + e^{-\frac{1}{2}t}}\left(2\sinh\left(\frac{t}{2}\right) + t\right) > 0.$$

Recall that $\|\Lambda_x(z)\|$ is the larger eigenvalue of $\Lambda_x(z)$, and $\left\|\Lambda_x(z)^{-1}\right\|$ is the inverse of the smallest eigenvalue of $\Lambda_x(z)$. For all $t > 0$ we have

$$0 < 1 - e^{-\frac{t}{2}} < 1 + e^{-\frac{t}{2}} < 2.$$

Besides,

$$\frac{1-e^{-t}-te^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}+\frac{1-e^{-t}+te^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}=2\left(1-\frac{te^{-t}}{1-e^{-t}}\right)<2,$$

and we just proved that both these terms are positive. Hence, each of them is smaller than 2. Thus, all the eigenvalues of $\Lambda_x(z)$ are smaller than 2 and $\|\Lambda_x(z)\|<2$.

For all $t>0$,

$$\frac{1-e^{-t}-te^{-\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}<1-e^{-\frac{t}{2}} \iff 1-e^{-t}-te^{-\frac{1}{2}t}<1-2e^{-\frac{t}{2}}+e^{-t} \iff 1-\frac{t}{2}<e^{-\frac{t}{2}},$$

and this is always true by convexity of the exponential. Thus, the smallest eigenvalue of $\Lambda_x(z)$ is $\dfrac{1-e^{-\|z\|^2}-\|z\|^2e^{-\frac{1}{2}\|z\|^2}}{1-e^{-\frac{1}{2}\|z\|^2}}=\dfrac{1}{f(\|z\|^2)}$, which proves our last claim. $\square$

Remark 3.4.18. To better understand the estimate (3.4.38), note that f is a decreasing function on $(0,+\infty)$. Moreover,

$$f(t)\xrightarrow[t\to+\infty]{}1, \qquad \text{and} \qquad f(t)\sim\frac{12}{t^2}$$

when t goes to 0.

Definition 3.4.19. For every $x\in M$, and $z\in T_xM\setminus\{0\}$, let $(L_x(0),L_x(z))$ be a centered Gaussian vector in $\mathbb{R}^2\otimes\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes T_x^*M$ with variance operator $\Lambda_x(z)$.

Recall that we defined the random vector $(X(t),Y(t))$ for all $t>0$ in the introduction (see Def. 3.1.4). Then $(X(t),Y(t))$ and $(L_x(0),L_x(z))$ are related as follows.

Lemma 3.4.20. *Let $x\in M$ and $z\in T_xM\setminus\{0\}$, then there exists an orthonormal basis of T_xM such that, for every orthonormal basis of $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$, the couple of $r\times n$ matrices associated to $(L_x(0),L_x(z))$ in these bases is distributed as $(X(\|z\|^2),Y(\|z\|^2))$.*

Proof. As in the proof of Lem. 3.4.15, let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$ of T_xM such that $z=\|z\|\frac{\partial}{\partial x_1}$. Let $(dx_1,\dots,dx_n)$ denote its dual basis. Let $(\zeta_1,\dots,\zeta_r)$ be any orthonormal basis of $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$, and let (e_1,e_2) denote the canonical basis of $\mathbb{R}^2$.

Then $z^*\otimes z=\|z\|^2\,dx_1\otimes\frac{\partial}{\partial x_1}$ and the matrix of the operator (3.4.35) in the orthonormal basis $(e_1\otimes dx_1,\dots,e_1\otimes dx_n,e_2\otimes dx_1,\dots,e_2\otimes dx_n)$ is:

$$\left(\begin{array}{cc|cc}
1-\frac{\|z\|^2e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0 & e^{-\frac{1}{2}\|z\|^2}\left(1-\frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0\\
0 & I_{n-1} & 0 & e^{-\frac{1}{2}\|z\|^2}I_{n-1}\\
\hline
e^{-\frac{1}{2}\|z\|^2}\left(1-\frac{\|z\|^2}{1-e^{-\|z\|^2}}\right) & 0 & 1-\frac{\|z\|^2e^{-\|z\|^2}}{1-e^{-\|z\|^2}} & 0\\
0 & e^{-\frac{1}{2}\|z\|^2}I_{n-1} & 0 & I_{n-1}
\end{array}\right), \tag{3.4.39}$$

where I_{n-1} stands for the identity matrix of size $n-1$. Since $\Lambda_x(z)$ equals this operator tensored by $\mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x}$, the matrix of $\Lambda_x(z)$ in the orthonormal basis:

$$\begin{aligned}(e_1\otimes dx_1\otimes\zeta_1,\dots,e_1\otimes dx_n\otimes\zeta_1,e_2\otimes dx_1\otimes\zeta_1,\dots,e_2\otimes dx_n\otimes\zeta_1,\\
e_1\otimes dx_1\otimes\zeta_2,\dots,e_2\otimes dx_n\otimes\zeta_2,\dots e_1\otimes dx_1\otimes\zeta_r,\dots,e_2\otimes dx_n\otimes\zeta_r)\end{aligned}$$

is exactly the variance matrix of $(X(\|z\|^2),Y(\|z\|^2))$ (cf. Def. 3.1.4).

Let us denote by $M_x(0)$ and $M_x(z)$ the matrices of $L_x(0)$ and $L_x(d)$ in the bases $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and $(\zeta_1, \dots, \zeta_r)$. Then $(M_x(0), M_x(z))$ is a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R})^2$. Moreover, we have just seen that the variance matrix of this random vector is the same as that of $(X(\|z\|^2), Y(\|z\|^2))$. This concludes the proof. $\square$

Corollary 3.4.21. *Let $x \in M$ and $z \in T_xM \setminus \{0\}$, then we have:*

$$\mathbb{E}\left[\left|\det^\perp(L_x(0))\right|\left|\det^\perp(L_x(z))\right|\right] = \mathbb{E}\left[\left|\det^\perp\left(X(\|z\|^2)\right)\right|\left|\det^\perp\left(Y(\|z\|^2)\right)\right|\right].$$

Proof. With the same notations as in the proof of Lemma 3.4.20 above, we have:

$$\mathbb{E}\left[\left|\det^\perp(M_x(0))\right|\left|\det^\perp(M_x(z))\right|\right] = \mathbb{E}\left[\left|\det^\perp\left(X(\|z\|^2)\right)\right|\left|\det^\perp\left(Y(\|z\|^2)\right)\right|\right],$$

since $(M_x(0), M_x(z))$ and $(X(\|z\|^2), Y(\|z\|^2))$ have the same distribution. Besides,

$$\left|\det^\perp(L_x(0))\right| = \left|\det^\perp(M_x(0))\right| \qquad \text{and} \qquad \left|\det^\perp(L_x(z))\right| = \left|\det^\perp(M_x(0))\right|.$$

$\square$

Let us now establish some facts about the distribution of $(X(t), Y(t))$ for $t > 0$.

Lemma 3.4.22. *For all $t > 0$, we have:*

$$\mathbb{E}\left[\left|\det^\perp(X(t))\right|\left|\det^\perp(Y(t))\right|\right] \leqslant n^r.$$

Proof. First, by the Cauchy–Schwarz inequality, we have:

$$\mathbb{E}\left[\left|\det^\perp(X(t))\right|\left|\det^\perp(Y(t))\right|\right] \leqslant \mathbb{E}\left[\left|\det^\perp(X(t))\right|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left|\det^\perp(Y(t))\right|^2\right]^{\frac{1}{2}}.$$

Then, the definition of $(X(t), Y(t))$ (Def. 3.1.4) shows that both $X(t)$ and $Y(t)$ are centered Gaussian vectors in $\mathcal{M}_{rn}(\mathbb{R})$ with variance matrix:

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & 0 \\ 0 & I_{n-1} \end{pmatrix} \otimes I_r. \tag{3.4.40}$$

in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^r$. Here I_r and I_{n-1} stand for the identity matrices of size r and $n-1$ respectively. Hence,

$$\mathbb{E}\left[\left|\det^\perp(X(t))\right|\left|\det^\perp(Y(t))\right|\right] \leqslant \mathbb{E}\left[\left|\det^\perp(X(t))\right|^2\right] = \mathbb{E}\left[\det\left(X(t)X(t)^{\mathrm{t}}\right)\right].$$

We denote by $X_1(t), \dots, X_r(t)$ the rows of $X(t)$. Then

$$X(t)X(t)^{\mathrm{t}} = (\langle X_i(t)\,, X_j(t)\rangle)_{1 \leqslant i,j \leqslant r}\,,$$

where we see $X_i(t)$ as an element of $\mathbb{R}^n$ and $\langle \cdot\,, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. Hence, $\det\left(X(t)X(t)^{\mathrm{t}}\right)$ is the Gram determinant of the family $(X_1(t), \dots, X_r(t))$, which is known to be the square of the r-dimensional volume of the parallelepiped spanned by these vectors. In particular,

$$\det\left(X(t)X(t)^{\mathrm{t}}\right) \leqslant \|X_1(t)\|^2 \cdots \|X_r(t)\|^2.$$

By (3.4.40), the $X_i(t)$ are independent identically distributed centered Gaussian vectors with variance matrix:

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

so that:

$$\det\left(X(t)X(t)^{\mathrm{t}}\right) \leqslant \mathbb{E}\Big[\|X_1(t)\|^2\cdots\|X_r(t)\|^2\Big] \leqslant \mathbb{E}\Big[\|X_1(t)\|^2\Big]^r = \left(n - \frac{te^{-t}}{1-e^{-t}}\right)^r \leqslant n^r.$$

□

Lemma 3.4.23. *We have the following estimate as $t \to +\infty$:*

$$\mathbb{E}\Big[\left|\det{}^{\perp}(X\,(t))\right|\left|\det{}^{\perp}(Y\,(t))\right|\Big] = (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2 + O\Big(te^{-\frac{t}{2}}\Big).$$

Proof. Let $(X(\infty), Y(\infty))$ be a standard Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R})^2 \simeq \mathbb{R}^{2nr}$, i.e. $X(\infty)$ and $Y(\infty)$ are independent standard Gaussian vectors in $\mathcal{M}_{rn}(\mathbb{R})$. Then,

$$\begin{aligned}
\mathbb{E}\Big[\left|\det{}^{\perp}(X(\infty))\right|\left|\det{}^{\perp}(Y(\infty))\right|\Big] &= \mathbb{E}\Big[\left|\det{}^{\perp}(X(\infty))\right|\Big]\,\mathbb{E}\Big[\left|\det{}^{\perp}(Y(\infty))\right|\Big]\\
&= \mathbb{E}\Big[\left|\det{}^{\perp}(X(\infty))\right|\Big]^2\\
&= (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2,
\end{aligned}$$

where we used (3.4.18) to get the last equality.

Then the proof is basically the same as that of Lemma 3.4.7. From Definition 3.1.4, we see that the variance operator $\Lambda(t)$ of $(X(t), Y(t))$ equals $\mathrm{Id} + O\Big(te^{-\frac{t}{2}}\Big)$ as $t \to +\infty$. Hence:

$$\det\left(\Lambda(t)\right) = 1 + O\Big(te^{-\frac{t}{2}}\Big) \qquad\text{and}\qquad \Lambda(t)^{-1} = \mathrm{Id} + O\Big(te^{-\frac{t}{2}}\Big).$$

Let $C > 0$ be such that $\left\|\Lambda(t)^{-1} - \mathrm{Id}\right\| \leqslant Cte^{-\frac{t}{2}}$. We denote by $L = (L_1, L_2)$ a generic element of $\mathcal{M}_{rn}(\mathbb{R})^2$ and by $\mathrm{d}L$ the normalized Lebesgue measure on this space. Then,

$$\begin{aligned}
&(2\pi)^{rn}\left|\det\left(\Lambda(t)\right)^{\frac{1}{2}}\mathbb{E}\Big[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\Big] - \mathbb{E}\Big[\left|\det{}^{\perp}(X(\infty))\right|\left|\det{}^{\perp}(Y(\infty))\right|\Big]\right|\\
&\qquad\leqslant \int \left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda(t)^{-1} - \mathrm{Id}\right)L\,,L\right\rangle\right) - 1\right| e^{-\frac{1}{2}\|L\|^2}\,\mathrm{d}L\\
&\qquad\leqslant \frac{C}{2}te^{-\frac{t}{2}}\int\left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\|L\|^2\exp\left(-\frac{1}{2}\left(1 - \frac{C}{2}te^{-\frac{t}{2}}\right)\|L\|^2\right)\mathrm{d}L\\
&\qquad= O\Big(te^{-\frac{t}{2}}\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\Big[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\Big]\\
&\qquad= \det\left(\Lambda(t)\right)^{-\frac{1}{2}}\left(\mathbb{E}\Big[\left|\det{}^{\perp}(X(\infty))\right|\left|\det{}^{\perp}(Y(\infty))\right|\Big] + O\Big(te^{-\frac{t}{2}}\Big)\right)\\
&\qquad= (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2 + O\Big(te^{-\frac{t}{2}}\Big).
\end{aligned}$$

□

Definition 3.4.24. Let $D_{n,r} : (0, +\infty) \to \mathbb{R}$ be the function defined by:

$$\forall t \in (0, +\infty), \qquad D_{n,r}(t) = \frac{\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right|\left|\det{}^{\perp}(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2.$$

Lemma 3.4.25. *We have:*

$$\int_0^{+\infty} |D_{n,r}(t)|\, t^{\frac{n-2}{2}}\, \mathrm{d}t < +\infty.$$

Proof. We first check the integrability of $|D_{n,r}(t)|\, t^{\frac{n-2}{2}}$ at $t = 0$. By Lemma 3.4.22, about $t = 0$ we have:

$$\begin{aligned}
|D_{n,r}(t)|\, t^{\frac{n-2}{2}} &\leqslant t^{\frac{n-2}{2}} \frac{\mathbb{E}\left[\left|\det^{\perp}(X(t))\right| \left|\det^{\perp}(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} + t^{\frac{n-2}{2}} (2\pi)^r \left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2 \\
&\leqslant t^{\frac{n-2}{2}} \frac{n^r}{(1-e^{-t})^{\frac{r}{2}}} + O(t^{\frac{n-2}{2}}) = O\Big(t^{\frac{n-2-r}{2}}\Big).
\end{aligned}$$

And this is integrable at $t = 0$ since $n - r \geqslant 1$.

Then, by Lemma 3.4.23, we have: $|D_{n,r}(t)|\, t^{\frac{n-2}{2}} = O\Big(t^{\frac{n}{2}} e^{-\frac{t}{2}}\Big)$ when t goes to infinity. This proves the integrability at infinity. □

Near-diagonal asymptotics for the correlated terms

The next step of the proof is to compute the contribution of the integral (3.4.10) on Δ_d. Let $R > 0$ be such that $2R$ is smaller than the injectivity radius of $\mathcal{X}$, as in Section 3.3. Let $d_3 \in \mathbb{N}$ be such that $\forall d \geqslant d_3$, $b_n \frac{\ln d}{\sqrt{d}} \leqslant R$. In the sequel we consider $d \geqslant \max(d_0, d_1, d_2, d_3)$.

Since we chose d large enough that $b_n \frac{\ln d}{\sqrt{d}} \leqslant R$ we can compute everything in the exponential chart about x. Let $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, we have:

$$\begin{aligned}
&\int_{\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 \\
&= \int_{x\in M} \left(\int_{y \in B_M\left(x, b_n \frac{\ln d}{\sqrt{d}}\right)} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M| \right) |\mathrm{d}V_M| \\
&= \int_{x\in M} \left(\int_{z \in B_{T_xM}\left(0, b_n \frac{\ln d}{\sqrt{d}}\right)} \phi_1(x)\phi_2(\exp_x(z))\mathcal{D}_d(x,\exp_x(z))\sqrt{\kappa(z)}\, \mathrm{d}z \right) |\mathrm{d}V_M|,
\end{aligned} \tag{3.4.41}$$

where $\sqrt{\kappa}$ is the density of $(\exp_x)^\star |\mathrm{d}V_M|$ with respect to the normalized Lebesgue measure on T_xM (see Sect. 3.3.2). Let $x \in M$, for all $z \in B_{T_xM}(0, b_n \ln d)$ we define

$$D_d(x,z) = \mathcal{D}_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right), \tag{3.4.42}$$

where $\mathcal{D}_d$ is defined by (3.4.9). Then, by a change of variable in (3.4.41),

$$\begin{aligned}
&\int_{\Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\, |\mathrm{d}V_M|^2 = \\
&d^{-\frac{n}{2}} \int_{x\in M} \left(\int_{z \in B_{T_xM}(0, b_n \ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_d(x,z) \left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{\frac{1}{2}} \mathrm{d}z \right) |\mathrm{d}V_M|,
\end{aligned} \tag{3.4.43}$$

and we need to compute the asymptotic of $D_d(x,z)$ as d goes to infinity. We start by computing $\left|\det^{\perp}\left(\mathrm{ev}^d_{x,y}\right)\right|$ when $(x,y) \in \Delta_d$.

Proposition 3.4.26. *Let $\alpha \in \left(0, \frac{1}{2r+1}\right)$, let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Then we have:*

$$\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)^*\right) = \left(1 - e^{-\|z\|^2}\right)^r \left(1 + O\big(d^{-\alpha}\big)\right), \tag{3.4.44}$$

where the error term does not depend on (x, z).

We will deduce Proposition 3.4.26 from the following two lemmas.

Lemma 3.4.27. *Let $\beta \in (0, 1)$ and $d \geqslant d_3$, then for every $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$, we have:*

$$\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right) = \left(1 - e^{-\|z\|^2}\right)^r + O\Big(d^{\beta-1}\Big),$$

where y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$. Moreover the error term depends on β but not on (x, z).

Lemma 3.4.28. *There exists $\widetilde{C} > 0$ such that, for all $\beta \in [0, 1)$, there exists $d_\beta \in \mathbb{N}$ such that: $\forall d \geqslant d_\beta$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,*

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)^*\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{C} d^{\beta-1},$$

where y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$.

Let us assume Lemmas 3.4.27 and 3.4.28 for now, and prove Prop. 3.4.26.

Proof of Proposition 3.4.26. First, note that if (3.4.44) holds for $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, then the same estimate holds for $z \in B_{T_xM}(0, b_n \ln d)$ since both sides of the equality vanish when $z = 0$. In the sequel we assume that $z \neq 0$.

Let $\alpha \in \left(0, \frac{1}{2r+1}\right)$, let $d \geqslant d_3$ and let $x \in M$. Then for any $z \in T_xM$ such that $\|z\| \geqslant d^{-\alpha}$, we have:

$$\left(1 - e^{-\|z\|^2}\right)^{-r} \leqslant \left(1 - \exp\left(-d^{-2\alpha}\right)\right)^{-r}. \tag{3.4.45}$$

Since $1 - e^{-t} = t\left(1 - \frac{t}{2} + O(t^2)\right)$ as $t \to 0$, there exists $\widetilde{C}_0$ such that for all $t \in (0, 1)$,

$$\left|\left(1 - e^{-t}\right)^{-r} - t^{-r}\right| \leqslant \widetilde{C}_0 t^{1-r}. \tag{3.4.46}$$

Hence, by (3.4.45), for any $d \geqslant d_3$, for any $x \in M$ and any $z \in T_xM$ such that $\|z\| \geqslant d^{-\alpha}$, we have:

$$\left(1 - e^{-\|z\|^2}\right)^{-r} \leqslant \left(d^{2r\alpha} + \widetilde{C}_0 d^{(2r-2)\alpha}\right) \leqslant d^{2r\alpha}\left(1 + \widetilde{C}_0\right). \tag{3.4.47}$$

Let $\beta = 1 - (2r + 1)\alpha$ and $\beta' = 1 - \alpha$, then β and $\beta' \in (0, 1)$. By Lemma 3.4.27, there exists $\widetilde{K}_\beta > 0$ such that: for all $d \geqslant d_3$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d)$,

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right) - \left(1 - e^{-\|z\|^2}\right)^r\right| \leqslant \widetilde{K}_\beta d^{\beta-1} = \widetilde{K}_\beta d^{-(2r+1)\alpha},$$

where $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Then, by (3.4.47), we have: $\forall d \geqslant d_3$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant d^{-\alpha} = d^{\beta'-1}$,

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d\left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{K}_\beta d^{-\alpha}\left(1 + \widetilde{C}_0\right),$$

Besides, let $d \geqslant d_{\beta'}$ and $x \in M$, then for all $z \in B_{T_xM}\left(0, d^{-\alpha}\right) \setminus \{0\}$ we have:

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant \widetilde{C} d^{-\alpha},$$

by Lemma 3.4.28. Finally, for all $d \geqslant \max(d_{\beta'}, d_3)$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, b_n \ln d\right) \setminus \{0\}$, we have:

$$\left|\left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)\right)\left(1 - e^{-\|z\|^2}\right)^{-r} - 1\right| \leqslant d^{-\alpha} \max\left(\widetilde{C}, 2\widetilde{K}_\beta\left(1 + \widetilde{C}_0\right)\right). \qquad \square$$

Proof of Lemma 3.4.27. Let $d \geqslant d_3$, let $x \in M$ and let $z \in B_{T_xM}(0, b_n \ln d)$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Since $\frac{\|z\|}{\sqrt{d}} < R$, let us write eq. (3.4.19) in the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ about x (see Sect. 3.3.1). We have:

$$\left(\frac{\pi}{d}\right)^n \mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^* = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}.$$

Then, by the near-diagonal estimates of Cor. 3.3.7, we have:

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} & e^{-\frac{1}{2}\|z\|^2}\left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-\frac{1}{2}} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} \\ e^{-\frac{1}{2}\|z\|^2}\left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-\frac{1}{2}} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} & \left(\kappa\left(\frac{z}{\sqrt{d}}\right)\right)^{-1} \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} \end{pmatrix} + O\left(\frac{(\ln d)^{2n+8}}{d}\right),$$

where the error term does not depend on (x, z). Recall that κ satisfies (3.3.6). Hence for all $z \in B(0, b_n \ln d)$,

$$\kappa\left(\frac{z}{\sqrt{d}}\right) = 1 + O\left(\frac{(\ln d)^2}{d}\right),$$

uniformly in x and z. Let $\beta \in (0,1)$, then we have:

$$\left(\frac{\pi}{d}\right)^n \mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^* = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right), \tag{3.4.48}$$

and the constant in the term $O\left(d^{\beta-1}\right)$ does not depend on (x, z). Since the dominant term on the right-hand side of (3.4.48) has bounded coefficients, we get the result by taking the determinant of (3.4.48). $\square$

Proof of Lemma 3.4.28. Let $d \geqslant \max(d_0, d_3)$ and let $x \in M$. Recall that $D^k_{(z,w)}$ denotes the k-th differential at (z, w) of a map from $T_x\mathcal{X} \times T_x\mathcal{X}$ to $\mathrm{End}\left(\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right)$.

The Chern connection reads $D + \mu_x$ in the real normal trivialization about x, where μ_x is a 1-form on $B_{T_x\mathcal{X}}(0, 2R)$. By definition of the real normal trivialization, $\mu_x(0) = 0$. Besides $\mu_x(z)$ is a smooth function of (x, z). Then, by compacity of M, there exists $K > 0$ such that $\|\mu_x(z)\| \leqslant K$ for all $x \in M$ and all $z \in B_{T_x\mathcal{X}}(0, R)$. Hence, there exists $K' > 0$ independent of x such that, for any smooth section S of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ over $B_{T_xM}(0, R) \times B_{T_xM}(0, R)$, we have:

$$\forall z, w \in B_{T_xM}(0, R), \qquad \left\|D_{(z,w)} S\right\| \leqslant K' \left\|S\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^1},$$

where $\|\cdot\|_{\mathcal{C}^1}$ was defined in Section 3.3.4. Since we use the exponential chart, we can argue similarly for the Levi–Civita connection. This gives a similar result for the higher derivatives of S. For all $k \in \mathbb{N}$, there exists $K_k > 0$ independent of x such that, for any smooth section S of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ over $B_{T_xM}(0,R) \times B_{T_xM}(0,R)$, we have:

$$\forall z, w \in B_{T_xM}(0,R), \qquad \left\|D^k_{(z,w)} S\right\| \leqslant K_k \left\|S\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^k}. \tag{3.4.49}$$

Since $d \geqslant d_0$, by eq. (3.4.49) and Thm. 3.3.9 we have: $\forall z, w \in B_{T_xM}(0,R)$,

$$\left\|D^2_{(z,w)} E_d\right\| \leqslant K_2 \left\|E_d\left(\exp_x(z), \exp_x(w)\right)\right\|_{\mathcal{C}^2} \leqslant C_2 K_2 d^{n+1}. \tag{3.4.50}$$

Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let us write eq. (3.4.19), in the real normal trivialization of $\mathcal{E}\otimes\mathcal{L}^d$ about x, as in the proof of Lem. 3.4.27. We have:

$$\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^* = \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}.$$

Then, by elementary operations on rows and columns,

$$\begin{aligned} \frac{1}{\|z\|^{2r}} \det\left(\mathrm{ev}^d_{x,y}\left(\mathrm{ev}^d_{x,y}\right)^*\right) &= \frac{1}{\|z\|^{2r}} \det\begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} \\ &= \det\begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \end{pmatrix} \end{pmatrix}. \end{aligned} \tag{3.4.51}$$

By Taylor's formula, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\left\|E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right)\right\| \leqslant \frac{\|z\|^2}{2d}\left(\sup_{w \in [0, \frac{z}{\sqrt{d}}]} \left\|D^2_{(0,w)} E_d\right\|\right). \tag{3.4.52}$$

Then, by (3.4.50), we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|}\left\|E_d\left(0, \frac{z}{\sqrt{d}}\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right)\right\| \leqslant \|z\| \, C_2 K_2 \pi^n. \tag{3.4.53}$$

Similarly, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|}\left\|E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d(0,0) - D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right)\right\| \leqslant \|z\| \, C_2 K_2 \pi^n. \tag{3.4.54}$$

A second order Taylor's formula gives:

$$\begin{aligned} \Bigg\|\left(E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0)\right) - \\ D^2_{(0,0)}E_d\left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right)\Bigg\| \leqslant \left(\frac{\|z\|}{\sqrt{d}}\right)^3 \left(\sup_{[0, \frac{z}{\sqrt{d}}]^2} \left\|D^3 E_d\right\|\right), \end{aligned}$$

and since $d \geqslant d_0$, by Thm. 3.3.9 and eq. (3.4.49) we have:

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|^2} \left\| \left(E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) - E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0) \right) - \right.$$
$$\left. D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right) \right\| \leqslant \|z\| \, C_3 K_3 \pi^n. \quad (3.4.55)$$

Finally, by equations (3.4.53), (3.4.54) and (3.4.55),

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0,\frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}},0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0)\end{pmatrix} \end{pmatrix}$$
$$= \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \\ \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) & \frac{1}{\|z\|^2} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right)\end{pmatrix} + O(\|z\|), \quad (3.4.56)$$

where the error term is uniform in x and d.

On the other hand, for every $x \in M$ and every $z \in T_xM \setminus \{0\}$, the diagonal estimates of Sect. 3.3.3 give (see (3.3.16)):

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|^2} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right) = \frac{\pi^n}{d^{n+1}} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\|z\|}\right)\left(\frac{z}{\|z\|}, 0\right)\right)$$
$$= \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O(d^{-1}),$$

where the error term is independent of x and z. Similarly,

$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) = \left(\frac{\pi}{d}\right)^n \frac{1}{\sqrt{d}} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\|z\|}\right) = O(d^{-1}),$$
$$\left(\frac{\pi}{d}\right)^n \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) = \left(\frac{\pi}{d}\right)^n \frac{1}{\sqrt{d}} D_{(0,0)}E_d \cdot \left(\frac{z}{\|z\|}, 0\right) = O(d^{-1}),$$

and

$$\left(\frac{\pi}{d}\right)^n E_d(0,0) = \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O(d^{-1}).$$

Thus

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(0, \frac{z}{\sqrt{d}}\right) \\ \frac{1}{\|z\|} D_{(0,0)}E_d \cdot \left(\frac{z}{\sqrt{d}}, 0\right) & \frac{1}{\|z\|^2} D^2_{(0,0)} E_d\left(\left(0, \frac{z}{\sqrt{d}}\right)\left(\frac{z}{\sqrt{d}}, 0\right)\right)\end{pmatrix} = \mathrm{Id} + O(d^{-1}), \quad (3.4.57)$$

where the error term is uniform in (x, z). By (3.4.56) and (3.4.57), there exist $\widetilde{C}_1$ and $\widetilde{C}_2 > 0$ such that we have: $\forall d \geqslant \max(d_0, d_3)$, $\forall x \in M$, $\forall z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$,

$$\left\| \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & \frac{1}{\|z\|}\left(E_d\left(0,\frac{z}{\sqrt{d}}\right) - E_d(0,0)\right) \\ \frac{1}{\|z\|}\left(E_d\left(\frac{z}{\sqrt{d}},0\right) - E_d(0,0)\right) & \frac{1}{\|z\|^2}\begin{pmatrix} E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) - E_d\left(\frac{z}{\sqrt{d}}, 0\right) \\ -E_d\left(0, \frac{z}{\sqrt{d}}\right) + E_d(0,0)\end{pmatrix} \end{pmatrix} - \mathrm{Id} \right\|$$
$$\leqslant \widetilde{C}_1 \|z\| + \widetilde{C}_2 \frac{1}{d}. \quad (3.4.58)$$

Let $\beta \in [0,1)$, then for all $d \geqslant \max(d_0, d_3)$, for all $x \in M$ and all $z \in B_{T_xM}\left(0, d^{\beta-1}\right)$, we have: $\widetilde{C}_1 \|z\| + \widetilde{C}_2 d^{-1} \leqslant d^{\beta-1}\left(\widetilde{C}_1 + \widetilde{C}_2\right)$. Let $d_\beta \in \mathbb{N}$ be such that $(d_\beta)^{\beta-1}\left(\widetilde{C}_1 + \widetilde{C}_2\right) \leqslant \frac{1}{2}$. Since the determinant is a smooth function, there exists $\widetilde{C}_3 > 0$ such that, for every operator Λ, if $\|\Lambda\| \leqslant \frac{1}{2}$, then $|\det(\mathrm{Id} + \Lambda) - 1| \leqslant \widetilde{C}_3 \|\Lambda\|$. Hence, by eq. (3.4.51) and (3.4.58), we have: for all $d \geqslant d_\beta$, for all $x \in M$, for all $z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,

$$\left| \frac{1}{\|z\|^{2r}} \left(\frac{\pi}{d}\right)^{2rn} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*\right) - 1 \right| \leqslant \left(\widetilde{C}_1 + \widetilde{C}_2\right) \widetilde{C}_3 d^{\beta-1}. \tag{3.4.59}$$

Recall that $\widetilde{C}_0$ was defined in the proof of Prop. 3.4.26 (see eq. (3.4.46)) and that, for all $x \in M$, for all $z \in B_{T_xM}(0,1) \setminus \{0\}$, we have:

$$\left| \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \leqslant \widetilde{C}_0 \|z\|^2 .$$

Then we have: $\forall d \geqslant d_\beta$, $\forall x \in M$, $\forall z \in B_{T_xM}\left(0, d^{\beta-1}\right) \setminus \{0\}$,

$$\begin{aligned}
&\left| \left(\frac{\pi}{d}\right)^{2nr} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*\right) \left(1 - e^{-\|z\|^2}\right)^{-r} - 1 \right| \\
&\quad = \left| \left(\frac{\pi}{d}\right)^{2nr} \frac{1}{\|z\|^{2r}} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*\right) \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \\
&\quad \leqslant \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} \left| \frac{1}{\|z\|^{2r}} \left(\frac{\pi}{d}\right)^{2rn} \det\left(\mathrm{ev}_{x,y}^d \left(\mathrm{ev}_{x,y}^d\right)^*\right) - 1 \right| + \left| \frac{\|z\|^{2r}}{\left(1 - e^{-\|z\|^2}\right)^r} - 1 \right| \\
&\quad \leqslant \left(1 + \widetilde{C}_0 d^{2\beta-2}\right) \left(\widetilde{C}_1 + \widetilde{C}_2\right) \widetilde{C}_3 d^{\beta-1} + \widetilde{C}_0 d^{2\beta-2} \\
&\quad \leqslant d^{\beta-1} \left(\left(\widetilde{C}_1 + \widetilde{C}_2\right) \widetilde{C}_3 \left(1 + \widetilde{C}_0\right) + \widetilde{C}_0\right) = d^{\beta-1} \widetilde{C},
\end{aligned}$$

where we define $\widetilde{C} > 0$ by the equality on the last line. $\square$

We now want to compute the limit of the conditional distribution of $\frac{\pi^n}{d^{n+1}}\left(\nabla_x^d s_d, \nabla_y^d s_d\right)$ given that $s_d(x) = 0 = s_d(y)$ for $(x,y) \in \Delta_d$. It is enough to compute the limit of $\Lambda_d(x,y)$ as $d \to +\infty$. Recall that Λ_d is defined by Def. 3.4.8. Since we work near the diagonal, we can write everything in the real normal trivialization centered at x (see Sect. 3.3.1).

Lemma 3.4.29. *Let $x \in M$ and let ∇^d be a real metric connection which is trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Let $\beta \in (0,1)$, then, in the real normal trivialization about x, we have: $\forall z \in B_{T_xM}(0, b_n \ln d)$,*

$$\begin{aligned}
&\frac{\pi^n}{d^{n+1}} \begin{pmatrix} \partial_x \partial_y^\sharp E_d(0,0) & \partial_x \partial_y^\sharp E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} = \\
&\quad \begin{pmatrix} \mathrm{Id}_{T_x^*M} & e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_x^*M} - z^* \otimes z\right) \\ e^{-\frac{1}{2}\|z\|^2}\left(\mathrm{Id}_{T_x^*M} - z^* \otimes z\right) & \mathrm{Id}_{T_x^*M} \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right),
\end{aligned}$$

where the error term does not depend on (x,z).

Proof. Let $x \in M$ and let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ of T_xM. We denote the corresponding coordinates on $T_xM \times T_xM$ by $(z_1, \dots, z_n, w_1, \dots, w_n)$ and by ∂_{z_i} and ∂_{w_j} the associated partial derivatives. Let $(dx_1, \dots, dx_n)$ denote the dual basis of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. By definition of ∇^d and $\partial_x \partial_y^\sharp E_d$ (see eq. (3.2.19)), for all $z, w \in B_{T_xM}(0, R)$, the matrix of $\partial_x \partial_y^\sharp E_d(z, w)$ in the orthonormal basis $(dx_1, \dots, dx_n)$ is:

$$\left(\partial_{z_i} \partial_{w_j} E_d(z, w)\right)_{1 \leqslant i,j \leqslant n}.$$

Note that this is a matrix with values in $\operatorname{End}\left(\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x\right)$. Recall that we defined the function ξ_d by (3.3.12). Then, by Cor. 3.3.7, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\partial_{z_i} \partial_{w_j} E_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) = \left(\frac{d}{\pi}\right)^n \partial_{z_i} \partial_{w_j} \xi_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) \operatorname{Id}_{\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x} + O\big((\ln d)^{2n+8}\big).$$

Then, eq. (3.3.15) shows that:

$$\begin{aligned}
\partial_{z_i} \partial_{w_j} \xi_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) &= \exp\left(-\frac{1}{2}\|z - w\|^2\right) \kappa\left(\frac{z}{\sqrt{d}}\right)^{-\frac{1}{2}} \kappa\left(\frac{w}{\sqrt{d}}\right)^{-\frac{1}{2}} \times \\
&\left(d\delta_{ij} - d(z_i - w_i)(z_j - w_j) - \frac{\sqrt{d}(z_j - w_j)\partial_{z_i}\kappa\left(\frac{z}{\sqrt{d}}\right)}{2\kappa\left(\frac{z}{\sqrt{d}}\right)} + \frac{\sqrt{d}(z_i - w_i)\partial_{w_j}\kappa\left(\frac{w}{\sqrt{d}}\right)}{2\kappa\left(\frac{w}{\sqrt{d}}\right)}\right) \\
&= d \exp\left(-\frac{1}{2}\|z - w\|^2\right) \left(\delta_{ij} - (z_i - w_i)(z_j - w_j)\right) + O\big((\ln d)^4\big),
\end{aligned}$$

where we used the fact that, uniformly in $z \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\kappa\left(\frac{z}{\sqrt{d}}\right) = 1 + O\left(\frac{(\ln d)^2}{d}\right)$$

$$\text{and } \forall i \in \{1, \dots, n\}, \qquad \partial_{z_i}\kappa\left(\frac{z}{\sqrt{d}}\right) = O\left(\frac{\ln d}{\sqrt{d}}\right).$$

Hence, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\begin{aligned}
&\frac{\pi^n}{d^{n+1}} \partial_{z_i} \partial_{w_j} E_d\left(\frac{z}{\sqrt{d}}, \frac{w}{\sqrt{d}}\right) = \\
&\qquad \exp\left(-\frac{1}{2}\|z - w\|^2\right) \left(\delta_{ij} - (z_i - w_i)(z_j - w_j)\right) \operatorname{Id}_{\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x} + O\left(\frac{(\ln d)^{2n+8}}{d}\right),
\end{aligned}$$

where the error term is independent of x, z and w. Furthermore, for any $\beta \in (0, 1)$, the term $O\left(\frac{(\ln d)^{2n+8}}{d}\right)$ can be replaced by $O\left(d^{\beta - 1}\right)$. Finally, for all $z, w \in B_{T_xM}(0, b_n \ln d)$, we have:

$$\begin{aligned}
&\frac{\pi^n}{d^{n+1}} \partial_x \partial_y^\sharp E_d(z, w) = \\
&\qquad \exp\left(-\frac{1}{2}\|z - w\|^2\right) \left(\operatorname{Id}_{T_x^*M} - (z - w)^* \otimes (z - w)\right) \otimes \operatorname{Id}_{\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x} + O\left(d^{\beta - 1}\right),
\end{aligned}$$

which yields the result. $\square$

A similar proof, using Cor. 3.3.7 and the expressions (3.3.13) and (3.3.14) for the partial derivatives of ξ_d yields the following.

Lemma 3.4.30. *Let $x \in M$ and let ∇^d be a real metric connection which is trivial over $B_{T_xM}(0,R)$ in the real normal trivialization about x. Let $\beta \in (0,1)$, then, in the real normal trivialization about x, we have: $\forall z \in B_{T_xM}(0, b_n \ln d)$,*

$$\frac{\pi^n}{d^{n+\frac{1}{2}}}\begin{pmatrix} \partial_x E_d(0,0) & \partial_x E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right)\end{pmatrix} = e^{-\frac{1}{2}\|z\|^2}\begin{pmatrix} 0 & z^* \\ -z^* & 0\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right)$$

$$\frac{\pi^n}{d^{n+\frac{1}{2}}}\begin{pmatrix} \partial_y^\sharp E_d(0,0) & \partial_y^\sharp E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right)\end{pmatrix} = e^{-\frac{1}{2}\|z\|^2}\begin{pmatrix} 0 & -z \\ z & 0\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right),$$

where $z^* \in T_x^*M$ is to be understood as the constant map $t \mapsto z^*$ from $\mathbb{R}$ to T_x^*M and $z \in T_xM$ is to be understood as the evaluation on z from T_x^*M to $\mathbb{R}$. Moreover, the error terms does not depend on (x,z).

We would like to get a similar asymptotic for the last term in the conditional variance operator (3.4.22), namely:

$$\begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right)\end{pmatrix}^{-1}.$$

Unfortunately, this term is singular on Δ, and this kills all hope to get a uniform estimate on $B_{T_xM}(0, b_n \ln d)\setminus\{0\}$. Instead, we obtain a uniform estimate on $B_{T_xM}(0, b_n \ln d)\setminus B_{T_xM}(0,\rho)$ for some $\rho > 0$. We need to carefully check how this estimate depends on ρ.

Lemma 3.4.31. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. Then, in the real normal trivialization about x, we have:*

$$\left(\frac{d}{\pi}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right)\end{pmatrix}^{-1} = \frac{1}{1-e^{-\|z\|^2}}\begin{pmatrix} 1 & -e^{-\frac{1}{2}\|z\|^2} \\ -e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)\right).$$

Here, the notation $O\left(\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}\right)$ means a quantity such that there exists $C>0$ and $\varepsilon>0$, independent of x, z, d and ρ, such that whenever $\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}} \leqslant \varepsilon$, the norm of this quantity is smaller than $C\frac{d^{\beta-1}}{1-e^{-\frac{1}{2}\rho^2}}$.

Proof. By eq. (3.4.19) and (3.4.48), we have:

$$\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right)\end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\beta-1}\right),$$

where the error term is independent of (x,z). Besides,

$$\left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1} = \frac{1}{1-e^{-\|z\|^2}}\begin{pmatrix} 1 & -e^{-\frac{1}{2}\|z\|^2} \\ -e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}, \tag{3.4.60}$$

and the eigenvalues of

$$\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x}$$

are $1 - e^{-\frac{1}{2}\|z\|^2}$ and $1 + e^{-\frac{1}{2}\|z\|^2}$, which shows that:

$$\left\| \left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} \right)^{-1} \right\| \leqslant \frac{1}{1 - e^{-\frac{1}{2}\|z\|^2}},$$

where $\|\cdot\|$ is the operator norm on $\mathrm{End}\left(\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$. Then, if $\|z\| \geqslant \rho$, we have:

$$\frac{1}{1 - e^{-\frac{1}{2}\|z\|^2}} \leqslant \frac{1}{1 - e^{-\frac{1}{2}\rho^2}}.$$

Thus,

$$\begin{aligned} &\left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} \left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1 - e^{-\frac{1}{2}\rho^2}}\right)\right). \end{aligned} \tag{3.4.61}$$

Taking the inverse of eq. (3.4.61), we get:

$$\begin{aligned} &\left(\frac{d}{\pi}\right)^n \begin{pmatrix} E_d(0,0) & E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ E_d\left(\frac{z}{\sqrt{d}}, 0\right) & E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix}^{-1} = \\ &\qquad \left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x} \right)^{-1} \left(\mathrm{Id} + O\left(\frac{d^{\beta-1}}{1 - e^{-\frac{1}{2}\rho^2}}\right)\right), \end{aligned}$$

where we used the mean value inequality and the fact that the differential of $\Lambda \mapsto \Lambda^{-1}$ is bounded from above on the closed ball of center Id and radius $\frac{1}{2}$. Finally, eq. (3.4.60) gives the result. ☐

Recall that $\Lambda_x(z)$ is defined for $x \in M$ and $z \in T_xM \setminus \{0\}$ by Def. 3.4.14. Recall also that $\Lambda_d(x, y)$ is defined by Def. 3.4.8.

Lemma 3.4.32. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then, in the real normal trivialization about x, we have:*

$$\Lambda_d(x, y) = \Lambda_x(z) + O\left(\frac{d^{\beta-1}}{(1 - e^{-\frac{1}{2}\rho^2})^2}\right),$$

where the constant in the error term does not depend on (x, z), d or ρ.

Proof. We know that $\Lambda_d(x, y)$ does not depend on the choice of ∇^d (see Rem.3.4.5). Hence, we can compute $\Lambda_d(x, y)$ with ∇^d trivial over $B_{T_xM}(0, R)$ in the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ about x.

Let $\beta \in (0,1)$ and $\rho \in (0,1)$, we apply Lemmas 3.4.30 and 3.4.31 for $\frac{\beta}{2}$. Then, in the real normal trivialization about x, we have:

$$\frac{\pi^n}{d^{n+1}} \begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix}$$
$$= \left(\begin{pmatrix} 0 & z^* \\ -z^* & 0 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\frac{\beta}{2}-1}\right)\right) \left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right)^{-1} \times$$
$$\left(\mathrm{Id} + O\left(\frac{d^{\frac{\beta}{2}-1}}{1-e^{-\frac{1}{2}\rho^2}} \right) \right) \left(\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(d^{\frac{\beta}{2}-1}\right)\right). \quad (3.4.62)$$

Since, $\rho \leqslant |z| < b_n \ln d$, the norm of

$$\left(\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right)^{-1}$$

is smaller than $\left(1-e^{-\frac{1}{2}\|z\|^2}\right)^{-1} \leqslant \left(1-e^{-\frac{1}{2}\rho^2}\right)^{-1}$, and the norms of the other matrices appearing in (3.4.62) are $O(\ln d)$. Hence, the expression (3.4.62) equals:

$$\frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} \begin{pmatrix} 0 & z^* \\ -z^* & 0 \end{pmatrix} \begin{pmatrix} 1 & -e^{-\frac{1}{2}\|z\|^2} \\ -e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2} \right)$$
$$= \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} \begin{pmatrix} z^*\otimes z & e^{-\frac{1}{2}\|z\|^2} z^*\otimes z \\ e^{-\frac{1}{2}\|z\|^2} z^*\otimes z & z^*\otimes z \end{pmatrix} \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2} \right), \quad (3.4.63)$$

where the error term is independent of (x,z). Finally, eq. (3.4.63) and Lemma 3.4.29 yield the result. □

Lemma 3.4.33. *Let $\beta \in (0,1)$ and $\rho \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant \rho$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then,*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d)=0\right] =$$
$$\mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right|\left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right] + O\left(f(\rho^2)^{\frac{r(n+1)}{2}+4} d^{\beta-1}\right),$$

where f is defined by Def. 3.4.16 and the constant in the error term does not depend on (x,z), d or ρ.

Proof. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, let $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$ then we have:

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d)=0\right]$$
$$= \mathbb{E}\left[\left|\det{}^\perp\left(\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d)=0\right]$$
$$= \mathbb{E}\left[\left|\det{}^\perp\left(L_d'(x)\right)\right|\left|\det{}^\perp\left(L_d'(y)\right)\right|\right],$$

where $(L_d'(x), L_d'(y))$ is a centered Gaussian vector in

$$\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_x^*M \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \otimes T_y^*M$$

with variance operator $\Lambda_d(x,y)$. We can consider $(L'_d(x), L'_d(y))$ as a random vector in $\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T_x^*M$, via the real normal trivialization about x. From now on, we work in this trivialization. Let $\rho \in (0,1)$ and $\beta \in (0,1)$, we assume that $\rho \leqslant \|z\| < b_n \ln d$. Then, by Lemma 3.4.32, we have:

$$\Lambda_d(x,y) = \Lambda_x(z) + O\left(\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right).$$

Moreover, by Cor. 3.4.17 and Rem. 3.4.18, $\left\|\Lambda_x(z)^{-1}\right\| \leqslant f\left(\|z\|^2\right) \leqslant f(\rho^2)$. Hence, we have:

$$\Lambda_d(x,y) = \Lambda_x(z)\left(\mathrm{Id} + O\left(f(\rho^2)\frac{d^{\beta-1}}{(1-e^{-\frac{1}{2}\rho^2})^2}\right)\right) = \Lambda_x(z)\left(\mathrm{Id} + O\left(f(\rho^2)^3 d^{\beta-1}\right)\right),$$

where we used the fact that $\frac{1}{1-e^{-\frac{1}{2}\rho^2}} \leqslant f(\rho^2)$ (see the proof of Cor. 3.4.17). Then, we get:

$$\det\left(\Lambda_d(x,y)\right) = \det\left(\Lambda_x(z)\right)\left(1 + O\left(f(\rho^2)^3 d^{\beta-1}\right)\right) \tag{3.4.64}$$

and

$$\Lambda_d(x,y)^{-1} = \Lambda_x(z)^{-1}\left(\mathrm{Id} + O\left(f(\rho^2)^3 d^{\beta-1}\right)\right) = \Lambda_x(z)^{-1} + O\left(f(\rho^2)^4 d^{\beta-1}\right).$$

Thus there exists $K>0$ and $\varepsilon>0$ such that, whenever $f(\rho^2)^4 d^{\beta-1} \leqslant \varepsilon$,

$$\left\|\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right\| \leqslant K f(\rho^2)^4 d^{\beta-1}.$$

By the mean value inequality, for every $L = (L_1, L_2) \in \mathbb{R}^2 \otimes T_x^*M \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$ we have:

$$\begin{aligned}&\left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right)L\,,L\right\rangle\right) - 1\right| \\ &\qquad\qquad \leqslant \frac{K}{2}\|L\|^2 f(\rho)^4 d^{\beta-1}\exp\left(\frac{K}{2}\|L\|^2 f(\rho)^4 d^{\beta-1}\right),\end{aligned}$$

whenever $f(\rho^2)^4 d^{\beta-1} \leqslant \varepsilon$. Let $\mathrm{d}L$ denote the normalized Lebesgue measure on this vector space, and recall that we defined $(L_x(0), L_x(z))$ above (Def. 3.4.19). Then, we have:

$$\begin{aligned}&(2\pi)^{nr}\left|\det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}}\mathbb{E}\left[\left|\det{}^{\perp}\left(L'_d(x)\right)\right|\left|\det{}^{\perp}\left(L'_d(y)\right)\right|\right]\right. \\ &\qquad\qquad \left. - \det\left(\Lambda_x(z)\right)^{\frac{1}{2}}\mathbb{E}\left[\left|\det{}^{\perp}(L_x(0))\right|\left|\det{}^{\perp}(L_x(z))\right|\right]\right| \\ &\leqslant \int\left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\exp\left(-\frac{1}{2}\left\langle\Lambda_x(z)^{-1}L\,,L\right\rangle\right)\times \\ &\qquad\qquad \left|\exp\left(-\frac{1}{2}\left\langle\left(\Lambda_d(x,y)^{-1} - \Lambda_x(z)^{-1}\right)L\,,L\right\rangle\right) - 1\right|\mathrm{d}L \\ &\leqslant \frac{K}{2}f(\rho^2)^4 d^{\beta-1}\int\left|\det{}^{\perp}(L_1)\right|\left|\det{}^{\perp}(L_2)\right|\|L\|^2 \times \\ &\qquad\qquad \exp\left(-\frac{1}{2}\left\langle\left(\Lambda_x(z)^{-1} - \frac{K}{2}f(\rho^2)^4 d^{\beta-1}\,\mathrm{Id}\right)L\,,L\right\rangle\right)\mathrm{d}L,\end{aligned}$$

whenever $f(\rho^2)^4 d^{\beta-1} \leqslant \varepsilon$. Since $\|\Lambda_x(d)\| < 2$ by Cor. 3.4.17, the smallest eigenvalue of $\Lambda_x(z)^{-1}$ is larger than $\frac{1}{2}$. Thus, if $f(\rho^2)^4 d^{\beta-1} \leqslant \frac{1}{2K}$, for every L we have:

$$\left\langle\left(\Lambda_x(z)^{-1} - \frac{K}{2}f(\rho^2)^4 d^{\beta-1}\,\mathrm{Id}\right)L\,,L\right\rangle \geqslant \frac{1}{4}\|L\|^2.$$

Hence, the last integral above is bounded by:

$$\int \left|\det{}^{\perp}(L_1)\right| \left|\det{}^{\perp}(L_2)\right| \|L\|^2 \exp\left(-\frac{1}{8}\|L\|^2\right) \mathrm{d}L < +\infty.$$

Then, we have:

$$\det\left(\Lambda_d(x,y)\right)^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right| \left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right] = \\ \det\left(\Lambda_x(z)\right)^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^{\perp}(L_x(0))\right| \left|\det{}^{\perp}(L_x(z))\right|\right] + O\Big(f(\rho^2)^4 d^{\beta-1}\Big),$$

and by (3.4.64), we obtain:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right| \left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right] = \\ \mathbb{E}\left[\left|\det{}^{\perp}(L_x(0))\right| \left|\det{}^{\perp}(L_x(z))\right|\right] \Big(1 + O\Big(f(\rho^2)^3 d^{\beta-1}\Big)\Big) \\ + \det\left(\Lambda_x(z)\right)^{-\frac{1}{2}} O\Big(f(\rho^2)^4 d^{\beta-1}\Big) \Big(1 + O\Big(f(\rho^2)^3 d^{\beta-1}\Big)\Big).$$

Since, for all $t > 0$ we have (see Lem. 3.4.15):

$$\frac{1}{1+e^{-\frac{1}{2}t}} \leqslant 1, \qquad \frac{1}{1-e^{-\frac{1}{2}t}} \leqslant f(t) \qquad \text{and} \qquad \frac{1+e^{-\frac{1}{2}t}}{1-e^{-t}+te^{-\frac{1}{2}t}} \leqslant f(t),$$

by Cor. 3.4.17 we have: $\det\left(\Lambda_x(z)\right)^{-\frac{1}{2}} \leqslant f(\rho^2)^{\frac{r(n+1)}{2}}$. Besides, by Cor. 3.4.21, we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}(L_x(0))\right| \left|\det{}^{\perp}(L_x(z))\right|\right] = \mathbb{E}\left[\left|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\right| \left|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\right|\right],$$

and by Lemma 3.4.22 this quantity is bounded from above by n^r. Finally, we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right| \left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right] = \mathbb{E}\left[\left|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\right| \left|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\right|\right] \\ + O\Big(f(\rho^2)^{4+\frac{r(n+1)}{2}} d^{\beta-1}\Big). \qquad \square$$

The following corollary is not necessary to the proof of Thm. 3.1.6 but is worth mentioning.

Corollary 3.4.34. *Let $\beta \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let ∇^d be any real metric connection. Then, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla_x^d s_d\right)\right| \left|\det{}^{\perp}\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d) = 0\right] = \\ \mathbb{E}\left[\left|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\right| \left|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\right|\right] + O\Big(d^{\beta-1}\Big),$$

where the error term depends on z but not on x.

Proof. Let us fix, β, x and z, then we set $\rho = \|z\|$ and we apply Lemma 3.4.33. $\square$

Before we can conclude the proof of Thm. 3.1.6, we need one last lemma.

Lemma 3.4.35. *Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. Let $\beta \in (0,1)$ and let ∇^d be any real metric connection. Then, we have:*

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla_x^d s_d\right)\right| \left|\det{}^{\perp}\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d) = 0\right] \leqslant \frac{(2r)!}{r!} n^r + O\Big(d^{\beta-1}\Big),$$

where the error term is independent of (x, z).

Proof. Let $x \in M$, let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ and let $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. As in the proof of Lem. 3.4.33, let $(L'_d(x), L'_d(y))$ be a centered Gaussian vector in $\mathbb{R}^2 \otimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \otimes T_x^*M$ which variance operator is $\Lambda_d(x,y)$, read in the real normal trivialization about x. In the sequel, we work in this trivialization. We have:

$$\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^\perp\left(\nabla_x^d s_d\right)\right|\left|\det{}^\perp\left(\nabla_y^d s_d\right)\right| \,\middle|\, \mathrm{ev}_{x,y}^d(s_d) = 0\right] = \mathbb{E}\left[\left|\det{}^\perp\left(L'_d(x)\right)\right| \left|\det{}^\perp\left(L'_d(y)\right)\right|\right].$$

The proof follows the same lines as that of Lem. 3.4.22, the main difference being that the variance operator is not explicit. An additional difficulty comes from the fact that the estimate for $\Lambda_d(x,y)$ given by Lemma 3.4.32 is not uniform in $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, hence it is useless here. Fortunately, we only need to bound its trace, which is bounded from above by that of the unconditional variance operator:

$$\frac{\pi^n}{d^{n+1}} \begin{pmatrix} \partial_x \partial_y^\sharp E_d(0,0) & \partial_x \partial_y^\sharp E_d\left(0, \frac{z}{\sqrt{d}}\right) \\ \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, 0\right) & \partial_x \partial_y^\sharp E_d\left(\frac{z}{\sqrt{d}}, \frac{z}{\sqrt{d}}\right) \end{pmatrix},$$

and Lemma 3.4.29 allows us to bound the latter.

By the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\det{}^\perp\left(L'_d(x)\right)\right| \left|\det{}^\perp\left(L'_d(y)\right)\right|\right] \leqslant \mathbb{E}\left[\left|\det{}^\perp\left(L'_d(x)\right)\right|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left|\det{}^\perp\left(L'_d(y)\right)\right|^2\right]^{\frac{1}{2}}. \tag{3.4.65}$$

Let $\Lambda_{d,1}(x,y)$ and $\Lambda_{d,2}(x,y)$ denote the variance operators of $L'_d(x)$ and $L'_d(y)$ respectively, so that:

$$\Lambda_d(x,y) = \begin{pmatrix} \Lambda_{d,1}(x,y) & * \\ * & \Lambda_{d,2}(x,y) \end{pmatrix}. \tag{3.4.66}$$

Let us choose orthonormal bases of T_xM and $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$. We denote by $(L'_d(x)_{ij})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n}}$ the coefficients of the matrix of $L'_d(x)$ in these bases, and by $(L'_d(x)_i)_{1 \leqslant i \leqslant r}$ its rows. As in the proof of Lem. 3.4.22, we have:

$$\begin{aligned} \left|\det{}^\perp\left(L'_d(x)\right)\right|^2 &= \det\left(L'_d(x)\left(L'_d(x)\right)^*\right) = \det\left(\left\langle L'_d(x)_i, L'_d(x)_j\right\rangle\right) \\ &\leqslant \left\|L'_d(x)_1\right\|^2 \cdots \left\|L'_d(x)_r\right\|^2. \end{aligned} \tag{3.4.67}$$

Then, we have:

$$\begin{aligned} \mathbb{E}\left[\left\|L'_d(x)_1\right\|^2 \cdots \left\|L'_d(x)_r\right\|^2\right] &= \mathbb{E}\left[\prod_{i=1}^r \left(\sum_{j=1}^n \left(L'_d(x)_{ij}\right)^2\right)\right] \\ &= \sum_{1 \leqslant j_1, \dots, j_r \leqslant n} \mathbb{E}\left[\prod_{i=1}^r \left(L'_d(x)_{i(j_i)}\right)^2\right]. \end{aligned} \tag{3.4.68}$$

Let $j_1, \dots, j_r \in \{1, \dots, r\}$, we denote $X_i = L'_d(x)_{i(j_i)}$. Then, by Wick's formula (see [TA07, lem. 11.6.1]), we have:

$$\mathbb{E}\left[\prod_{i=1}^r \left(L'_d(x)_{i(j_i)}\right)^2\right] = \mathbb{E}\left[\prod_{i=1}^r (X_i)^2\right] = \sum_{(\{a_i,b_i\})} \prod_{i=1}^r \mathbb{E}\left[X_{\lfloor \frac{a_i}{2} \rfloor} X_{\lfloor \frac{b_i}{2} \rfloor}\right],$$

where we sum over all the partitions into pairs $(\{a_i,b_i\})_{1\leqslant i\leqslant r}$ of $\{1,\dots,2r\}$. Hence, by Cauchy-Schwarz inequality again, we get:

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{r}\left(L_d'(x)_{i(j_i)}\right)^2\right] &\leqslant \sum_{(\{a_i,b_i\})}\prod_{i=1}^{r}\mathbb{E}\left[\left(X_{\lfloor\frac{a_i}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\left(X_{\lfloor\frac{b_i}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\\
&\leqslant \sum_{(\{a_i,b_i\})}\prod_{k=1}^{2r}\mathbb{E}\left[\left(X_{\lfloor\frac{k}{2}\rfloor}\right)^2\right]^{\frac{1}{2}}\\
&\leqslant \sum_{(\{a_i,b_i\})}\prod_{l=1}^{r}\mathbb{E}\left[\left(X_l\right)^2\right]\\
&\leqslant \frac{(2r)!}{2^r r!}\prod_{i=1}^{r}\mathbb{E}\left[\left(L_d'(x)_{i(j_i)}\right)^2\right].
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\sum_{1\leqslant j_1,\dots,j_r\leqslant n}\mathbb{E}\left[\prod_{i=1}^{r}\left(L_d'(x)_{i(j_i)}\right)^2\right] &\leqslant \frac{(2r)!}{2^r r!}\sum_{1\leqslant j_1,\dots,j_r\leqslant n}\prod_{i=1}^{r}\mathbb{E}\left[\left(L_d'(x)_{i(j_i)}\right)^2\right]\\
&\leqslant \frac{(2r)!}{2^r r!}\prod_{i=1}^{r}\left(\sum_{j=1}^{n}\mathbb{E}\left[\left(L_d'(x)_{ij}\right)^2\right]\right)\\
&\leqslant \frac{(2r)!}{2^r r!}\left(\sum_{i=1}^{r}\sum_{j=1}^{n}\mathbb{E}\left[\left(L_d'(x)_{ij}\right)^2\right]\right)^r\\
&\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^r,
\end{aligned}\tag{3.4.69}$$

where Tr stands for the trace operator. Finally, by (3.4.67), (3.4.68) and (3.4.69), we have:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right|^2\right]\leqslant\frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^r,$$

and similarly,

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(y)\right)\right|^2\right]\leqslant\frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,2}(x,y)\right)^r.$$

Thus, by (3.4.65), we get:

$$\begin{aligned}
\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d'(x)\right)\right|\left|\det{}^{\perp}\left(L_d'(y)\right)\right|\right] &\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d,1}(x,y)\right)^{\frac{r}{2}}\operatorname{Tr}\left(\Lambda_{d,2}(x,y)\right)^{\frac{r}{2}}\\
&\leqslant \frac{(2r)!}{2^r r!}\operatorname{Tr}\left(\Lambda_{d}(x,y)\right)^r.
\end{aligned}\tag{3.4.70}$$

Let $\beta\in(0,1)$, by eq. (3.4.70), we only need to prove that $\operatorname{Tr}\left(\Lambda_d(x,y)\right)\leqslant 2n+O\left(d^{\beta-1}\right)$ to complete the proof. By eq. (3.4.19),

$$\begin{pmatrix}E_d(x,x) & E_d(x,y)\\ E_d(y,x) & E_d(y,y)\end{pmatrix}$$

is a variance operator. Hence it is a positive symmetric operator and so is its inverse. Besides, by (3.4.21), we know that:

$$\begin{pmatrix}\partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y)\\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y)\end{pmatrix}=\begin{pmatrix}\partial_x E_d(x,x) & \partial_x E_d(x,y)\\ \partial_x E_d(y,x) & \partial_x E_d(y,y)\end{pmatrix}^*.$$

Then, the diagonal coefficients of:

$$\begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix}$$

are non-negative, and so is its trace. Finally, by the definition of $\Lambda_d(x,y)$ (Def. 3.4.8), we have:

$$\operatorname{Tr}\left(\Lambda_d(x,y)\right) \leqslant \frac{\pi^n}{d^{n+1}} \operatorname{Tr}\begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y) \\ \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix}. \tag{3.4.71}$$

Note that what we have done so far works for any choice of connection since $\Lambda_d(x,y)$ is independent of this choice. However, the right-hand side of eq. (3.4.71) depends on the choice ∇^d. We use a real metric connection that is trivial on $B_{T_xM}(0,R)$ in the real normal trivialization about x. Then, by Lemma 3.4.29, we have:

$$\operatorname{Tr}\left(\Lambda_d(x,y)\right) \leqslant 2n + O\left(d^{\beta-1}\right). \qquad \square$$

Conclusion of the proof

We can now prove Theorem 3.1.6.

Lemma 3.4.36. *Let $\alpha > 0$, let $\phi \in \mathcal{C}^0(M)$ and let $x \in M$, then we have:*

$$\left|\int_{B_{T_xM}(0,d^{-\alpha})} \phi\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\left(\frac{1}{d^r}D_d(x,z) - D_{n,r}(\|z\|^2)\right)\mathrm{d}z\right| = \|\phi\|_\infty O\left(d^{(r-n)\alpha}\right),$$

where the error term does not depend on x or ϕ.

Proof. We have:

$$\begin{aligned}\left|\int_{B_{T_xM}(0,d^{-\alpha})} \phi\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\left(\frac{1}{d^r}D_d(x,z) - D_{n,r}(\|z\|^2)\right)\mathrm{d}z\right| \\ \leqslant \|\phi\|_\infty\left(\sup_{B_{T_xM}\left(0,b_n\frac{\ln d}{\sqrt{d}}\right)}|\kappa|^{\frac{1}{2}}\right)\int_{B_{T_xM}(0,d^{-\alpha})}\left(\frac{1}{d^r}\left|D_d(x,z)\right| + \left|D_{n,r}(\|z\|^2)\right|\right)\mathrm{d}z.\end{aligned}$$

Since $\kappa(z) = 1 + O\left(\|z\|^2\right)$ uniformly in x (see (3.3.6)), we have:

$$\sup_{B_{T_xM}\left(0,b_n\frac{\ln d}{\sqrt{d}}\right)}|\kappa|^{\frac{1}{2}} = 1 + O\left(\frac{(\ln d)^2}{d}\right),$$

and this term is bounded. Thus, we only need to consider the integrals of $\frac{1}{d^r}\left|D_d(x,z)\right|$ and $\left|D_{n,r}(\|z\|^2)\right|$. By Lemma 3.4.22, we have:

$$\begin{aligned}\int_{B(0,d^{-\alpha})}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z \leqslant \operatorname{Vol}\left(\mathbb{S}^{n-1}\right)\int_{\rho=0}^{d^{-\alpha}}\frac{\mathbb{E}\left[\left|\det^\perp\left(X(\rho^2)\right)\right|\left|\det^\perp\left(Y(\rho^2)\right|\right]}{\left(1-e^{-\rho^2}\right)^{\frac{r}{2}}}\rho^{n-1}\,\mathrm{d}\rho \\ + (2\pi)^r\left(\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r}\right)}{\operatorname{Vol}\left(\mathbb{S}^n\right)}\right)^2\operatorname{Vol}\left(B_{T_xM}\left(0,d^{-\alpha}\right)\right) \\ \leqslant \frac{n^r}{2}\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{d^{-2\alpha}}\frac{t^{\frac{n-2}{2}}}{(1-e^{-t})^{\frac{r}{2}}}\,\mathrm{d}t + O\left(d^{-n\alpha}\right).\end{aligned} \tag{3.4.72}$$

Then, since there exists $C > 0$ such that $\frac{t}{1-e^{-t}} \leqslant C$ for all $t \in (0, 1]$, we get:

$$\int_{t=0}^{d^{-2\alpha}} \frac{t^{\frac{n-2}{2}}}{(1-e^{-t})^{\frac{r}{2}}} \, \mathrm{d}t \leqslant C \int_{t=0}^{d^{-2\alpha}} t^{\frac{n-2-r}{2}} \, \mathrm{d}t = O\Big(d^{(r-n)\alpha}\Big). \tag{3.4.73}$$

Hence, $\displaystyle\int_{B(0,d^{-\alpha})} \left|D_{n,r}(\|z\|^2)\right| \mathrm{d}z = O\Big(d^{(r-n)\alpha}\Big)$. Let us denote $y = \exp_x\left(\frac{z}{\sqrt{d}}\right)$. By the definition of $D_d(x,z)$ (cf. (3.4.42)), we have:

$$\begin{aligned} \frac{1}{d^r} \left|D_d(x,z)\right| \leqslant{}& \frac{1}{d^r} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_x^d s_d\right)\right|\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \Big| \mathrm{ev}_{x,y}^d(s_d) = 0\Big]}{\left|\det^\perp\left(\mathrm{ev}_{x,y}^d\right)\right|} \\ &+ \frac{1}{d^r} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_x^d s_d\right)\right| \Big| s_d(x) = 0\Big]}{\left|\det^\perp(\mathrm{ev}_x^d)\right|} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \Big| s_d(y) = 0\Big]}{\left|\det^\perp\left(\mathrm{ev}_y^d\right)\right|}. \end{aligned}$$

Then, let $\beta \in (0,1)$ and $\beta' \in \left(0, \frac{1}{2r+1}\right)$, by Prop. 3.4.26 and Lem. 3.4.35 we have:

$$\begin{aligned} \frac{1}{d^r} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_x^d s_d\right)\right|\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \Big| \mathrm{ev}_{x,y}^d(s_d) = 0\Big]}{\left|\det^\perp\left(\mathrm{ev}_{x,y}^d\right)\right|} &\leqslant \frac{\frac{(2r)!}{r!} n^r + O\big(d^{\beta-1}\big)}{\Big(1 - e^{-\|z\|^2}\Big)^{\frac{r}{2}}} \Big(1 + O\Big(d^{-\beta'}\Big)\Big) \\ &\leqslant C \left(\frac{1}{1 - e^{-\|z\|^2}}\right)^{\frac{r}{2}}, \end{aligned}$$

for some large C. By a polar change of coordinates similar to (3.4.72) and (3.4.73), we show that the integral of this term over $B_{T_xM}(0, d^{-\alpha})$ is a $O\big(d^{(r-n)\alpha}\big)$. Finally, by Lem. 3.4.6 and 3.4.7 we have:

$$\frac{1}{d^r} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_x^d s_d\right)\right| \Big| s_d(x) = 0\Big]}{\left|\det^\perp(\mathrm{ev}_x^d)\right|} \frac{\mathbb{E}\Big[\left|\det^\perp\left(\nabla_y^d s_d\right)\right| \Big| s_d(y) = 0\Big]}{\left|\det^\perp\left(\mathrm{ev}_y^d\right)\right|} = O(1).$$

Hence the integral of this term over $B_{T_xM}(0, d^{-\alpha})$ is a $O(d^{-n\alpha})$. □

Recall that we defined $\alpha_0 = \dfrac{n-r}{2(2r+1)(2n+1)}$ (see Ntn. 3.1.5). Let us also denote $\alpha_1 = \dfrac{\alpha_0}{n-r} = \dfrac{1}{2(2r+1)(2n+1)}$.

Lemma 3.4.37. *Let $\alpha \in (0, \alpha_1)$, let $\phi \in \mathcal{C}^0(M)$ and $x \in M$, then we have:*

$$\begin{aligned} \left|\int_{d^{-\alpha} \leqslant \|z\| < b_n \ln d} \phi\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) \kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}} \left(\frac{1}{d^r} D_d(x,z) - D_{n,r}(\|z\|^2)\right) \mathrm{d}z\right| \\ = \|\phi\|_\infty O\Big(d^{(r-n)\alpha}\Big), \end{aligned}$$

where the error term does not depend on x or ϕ.

Proof. As in the proof of Lemma 3.4.36, since $\kappa^{\frac{1}{2}}$ is bounded on $B_{T_xM}\left(0, b_n \frac{\ln d}{\sqrt{d}}\right)$ uniformly in $x \in M$, we only need to prove that:

$$\left|\frac{1}{d^r} D_d(x,z) - D_{n,r}(\|z\|^2)\right| = O\Big(d^{(r-n)\alpha - \alpha'}\Big)$$

for some $\alpha' > 0$. Then, since $\operatorname{Vol}(B_{T_xM}(0, b_n \ln d)) = O((\ln d)^n) = O\left(d^{\alpha'}\right)$, we get the result by integrating over $B_{T_xM}(0, b_n \ln d) \setminus B_{T_xM}(0, d^{-\alpha})$.

Since $\alpha \in (0, \alpha_1)$, we have $0 < n\alpha < \frac{1}{2r+1}$ and we can choose a positive $\beta \in \left(n\alpha, \frac{1}{2r+1}\right)$. Let $\beta' \in (0,1)$ be such that:

$$1 - 2\alpha(8 + r(n+1)) - \beta < \beta' < 1 - 2\alpha(8 + r(n+1)) - n\alpha < 1. \tag{3.4.74}$$

We already know that $-\beta < -n\alpha$, so we only need to check that $0 < 1 - \alpha(16 + 2rn + 2r + n)$ to ensure the existence of such a β'. This goes as follows:

$$1 - \alpha(16 + 2rn + 2r + n) > 1 - 2\alpha_1(8 + rn + n + r) = \frac{3rn + n + r - 7}{(2r+1)(2n+1)} > 0.$$

By Lemma 3.4.33, for every $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$ such that $\|z\| \geqslant d^{-\alpha}$ we have:

$$\begin{aligned}\left(\frac{\pi^n}{d^{n+1}}\right)^r \mathbb{E}\left[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right] &= \\ \mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right|\left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right] &+ O\left(f(d^{-2\alpha})^{\frac{r(n+1)}{2}+4} d^{\beta'-1}\right),\end{aligned} \tag{3.4.75}$$

where, as usual, y stands for $\exp_x\left(\frac{z}{\sqrt{d}}\right)$. Recall that we have: $f(t) \sim \frac{12}{t^2}$ as $t \to 0$ (cf. Rem. 3.4.18). Then, we get:

$$f(d^{-2\alpha})^{\frac{r(n+1)}{2}+4} = O\left(d^{2\alpha(8+r(n+1))}\right).$$

We set $\alpha' = 1 - 2\alpha\,(8 + r(n+1)) - \beta' - n\alpha$, so that the error term in (3.4.75) is a $O\left(d^{-n\alpha-\alpha'}\right)$. By (3.4.74), we have $\alpha' > 0$.

By Prop. 3.4.26, applied for β, and eq. (3.4.75) we have:

$$\begin{aligned}\frac{1}{d^r}\frac{\mathbb{E}\left[\left|\det{}^\perp(\nabla^d_x s_d)\right|\left|\det{}^\perp(\nabla^d_y s_d)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right]}{\left|\det{}^\perp(\mathrm{ev}^d_{x,y})\right|} &= \\ \frac{\mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right|\left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right] + O\left(d^{-n\alpha-\alpha'}\right)}{\left(1 - e^{-\|z\|^2}\right)^{\frac{r}{2}}}&\left(1 + O\left(d^{-\beta}\right)\right),\end{aligned} \tag{3.4.76}$$

for all $x \in M$ and $z \in T_xM$ such that $d^{-\alpha} \leqslant \|z\| < b_n \ln d$. Since

$$\left(1 - e^{-d^{-2\alpha}}\right)^{-\frac{r}{2}} = O(d^{r\alpha}),$$

and the numerator of (3.4.76) is bounded (cf. Lemma 3.4.22), the right-hand side of equation (3.4.76) equals:

$$\frac{\mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right|\left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right]}{\left(1 - e^{-\|z\|^2}\right)^{\frac{r}{2}}} + O\left(d^{(r-n)\alpha-\alpha'}\right) + O\left(d^{r\alpha-\beta}\right).$$

Moreover, $n\alpha + \alpha' = 1 - 2\alpha\,(8 + r(n+1)) - \beta' < \beta$ (see eq. (3.4.74)), so that we have:

$$\begin{aligned}\frac{1}{d^r}\frac{\mathbb{E}\left[\left|\det{}^\perp(\nabla^d_x s_d)\right|\left|\det{}^\perp(\nabla^d_y s_d)\right| \,\middle|\, \mathrm{ev}^d_{x,y}(s_d) = 0\right]}{\left|\det{}^\perp(\mathrm{ev}^d_{x,y})\right|} &= \\ \frac{\mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right|\left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right]}{\left(1 - e^{-\|z\|^2}\right)^{\frac{r}{2}}} &+ O\left(d^{(r-n)\alpha-\alpha'}\right).\end{aligned}$$

On the other hand, by Lemmas 3.4.6 and 3.4.7, we have:

$$\frac{1}{d^r}\frac{\mathbb{E}\left[\left|\det^{\perp}(\nabla^d_x s_d)\right| \,\middle|\, s_d(x)=0\right]}{\left|\det^{\perp}(\mathrm{ev}^d_x)\right|}\frac{\mathbb{E}\left[\left|\det^{\perp}(\nabla^d_y s_d)\right| \,\middle|\, s_d(y)=0\right]}{\left|\det^{\perp}(\mathrm{ev}^d_y)\right|} = (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\right)^2 + O\big(d^{-1}\big).$$

Once again, eq. (3.4.74) shows that $n\alpha+\alpha'<\beta<1$. A fortiori $(n-r)\alpha+\alpha'<1$. Thus, for all $x\in M$ and $z\in T_xM$ such that $d^{-\alpha}\leqslant \|z\|<b_n\ln d$, we have:

$$\left|\frac{1}{d^r}D_d(x,z)-D_{n,r}(\|z\|^2)\right| = O\left(d^{(r-n)\alpha-\alpha'}\right),$$

where $\alpha'>0$ and the error term is independent of (x,z). □

Proposition 3.4.38. *Let $\alpha\in(0,\alpha_0)$, let ϕ_1 and $\phi_2\in\mathcal{C}^0(M)$, we have the following asymptotic as $d\to+\infty$:*

$$\begin{aligned}\frac{1}{d^r}\int_{x\in M}&\left(\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z\right)|\mathrm{d}V_M|\\ &=\int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_{n,r}(\|z\|^2)\,\mathrm{d}z\right)|\mathrm{d}V_M|\\ &\qquad+\|\phi_1\|_\infty\|\phi_2\|_\infty O\big(d^{-\alpha}\big),\end{aligned}$$

where the error term does not depend on (ϕ_1,ϕ_2).

Proof. Let $\alpha\in(0,\alpha_0)$, we set $\alpha'=\frac{\alpha}{n-r}\in(0,\alpha_1)$. Let $\phi_1,\phi_2\in\mathcal{C}^0(M)$ and let $x\in M$, we apply Lemmas 3.4.36 and 3.4.37 for α' and ϕ_2. Then, we have:

$$\begin{aligned}\frac{1}{d^r}\int_{z\in B_{T_xM}(0,b_n\ln d)}&\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z\\ &=\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_{n,r}(\|z\|^2)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z\\ &\qquad+|\phi_1(x)|\,\|\phi_2\|_\infty O\left(d^{(r-n)\alpha'}\right),\end{aligned}\tag{3.4.77}$$

and the error term can be rewritten as $O(d^{-\alpha})$.

Since $\kappa(z)^{\frac{1}{2}}=1+O\left(\|z\|^2\right)$ (cf. (3.3.6)), there exists $C>0$ independent of x such that for all $z\in B_{T_xM}(0,R)$, $\left|\kappa(z)^{\frac{1}{2}}-1\right|\leqslant C\|z\|^2$. Then, we get:

$$\begin{aligned}&\left|\int_{z\in B_{T_xM}(0,b_n\ln d)}\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)D_{n,r}(\|z\|^2)\left(\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}-1\right)\mathrm{d}z\right|\\ &\qquad\leqslant|\phi_1(x)|\,\|\phi_2\|_\infty C\frac{(b_n\ln d)^2}{d}\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z\\ &\qquad\leqslant|\phi_1(x)|\,\|\phi_2\|_\infty\frac{C}{2}\frac{(b_n\ln d)^2}{d}\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{(b_n\ln d)^2}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t.\end{aligned}$$

Since $|D_{n,r}(t)|\,t^{\frac{n-2}{2}}$ is integrable on $(0,+\infty)$ (Lem. 3.4.25) and $\alpha<1$, we have:

$$\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z = \\ \int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z + |\phi_1(x)|\,\|\phi_2\|_\infty\, O\big(d^{-\alpha}\big), \tag{3.4.78}$$

where the error term in independent of x. By (3.4.77) and (3.4.78), we have:

$$\frac{1}{d^r}\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_d(x,z)\kappa\left(\frac{z}{\sqrt{d}}\right)^{\frac{1}{2}}\mathrm{d}z = \\ \int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z + |\phi_1(x)|\,\|\phi_2\|_\infty\, O\big(d^{-\alpha}\big),$$

uniformly in $x\in M$. Integrating this relation over M yields the result. □

Now, let $\alpha\in(0,\alpha_0)$, let ϕ_1 and $\phi_2\in\mathcal{C}^0(M)$, then by eq. (3.4.10), Prop. 3.4.12, eq. (3.4.43) and Prop. 3.4.38 we have:

$$\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2) = \\ \frac{d^{r-\frac{n}{2}}}{(2\pi)^r}\int_{x\in M}\left(\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z\right)|\mathrm{d}V_M| \\ + \|\phi_1\|_\infty\,\|\phi_2\|_\infty\, O\Big(d^{r-\frac{n}{2}-\alpha}\Big), \tag{3.4.79}$$

where the error term is independent of (ϕ_1,ϕ_2). Then, we have:

$$\left|\int_{z\in B_{T_xM}(0,b_n\ln d)}\left(\phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right)-\phi_1(x)\phi_2(x)\right)D_{n,r}(\|z\|^2)\,\mathrm{d}z\right| \\ \leqslant \|\phi_1\|_\infty\,\varpi_{\phi_2}\left(\frac{b_n\ln d}{\sqrt{d}}\right)\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z,$$

where ϖ_{ϕ_2} is the continuity modulus of ϕ_2 (see Def. 3.1.2). Besides, by a polar change of coordinates, we have:

$$\int_{z\in B(0,b_n\ln d)}\left|D_{n,r}(\|z\|^2)\right|\mathrm{d}z = \frac{1}{2}\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{(b_n\ln d)^2}|D_{n,r}(t)|\,t^{\frac{n-2}{2}}\,\mathrm{d}t, \tag{3.4.80}$$

and this quantity is bounded, by Lemma 3.4.25. Then,

$$\int_{z\in B_{T_xM}(0,b_n\ln d)} \phi_1(x)\phi_2\left(\exp_x\left(\frac{z}{\sqrt{d}}\right)\right) D_{n,r}(\|z\|^2)\,\mathrm{d}z = \\ \phi_1(x)\phi_2(x)\int_{z\in B_{T_xM}(0,b_n\ln d)} D_{n,r}(\|z\|^2)\,\mathrm{d}z + \|\phi_1\|_\infty\,\varpi_{\phi_2}\left(\frac{b_n\ln d}{\sqrt{d}}\right)O(1), \tag{3.4.81}$$

where the error term is independent of (ϕ_1,ϕ_2).

Let $\beta \in \left(0, \frac{1}{2}\right)$, then there exists $C_\beta > 0$ such that for all $d \in \mathbb{N}^*$, $b_n \frac{\ln d}{\sqrt{d}} \leqslant C_\beta d^{-\beta}$. Since ϖ_{ϕ_2} is a non-decreasing function, we have $\varpi_{\phi_2}\left(b_n \frac{\ln d}{\sqrt{d}}\right) \leqslant \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right)$. By (3.4.79), (3.4.80) and (3.4.81), we obtain:

$$\begin{aligned}\mathrm{Var}(|\mathrm{d}V_d|)\left(\phi_1, \phi_2\right) = &\\ & d^{r-\frac{n}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \left(\int_M \phi_1 \phi_2 \left|\mathrm{d}V_M\right|\right) \left(\frac{1}{2} \int_{t=0}^{(b_n \ln d)^2} D_{n,r}(t) t^{\frac{n-2}{2}} \,\mathrm{d}t\right) \\ & + \left\|\phi_1\right\|_\infty \left\|\phi_2\right\|_\infty O\!\left(d^{r-\frac{n}{2}-\alpha}\right) + \left\|\phi_1\right\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\!\left(d^{r-\frac{n}{2}}\right). \quad (3.4.82)\end{aligned}$$

By Lemma 3.4.23, we have: $\left|D_{n,r}(t)\right| = O\!\left(t e^{-\frac{t}{2}}\right)$. Then there exists some $C > 0$ such that, for all t large enough,

$$\left|D_{n,r}(t)\right| t^{\frac{n-2}{2}} \leqslant C e^{-\frac{t}{4}}.$$

Then, for d large enough we have:

$$\left|\int_{t=(b_n \ln d)^2}^{+\infty} D_{n,r}(t) t^{\frac{n-2}{2}} \,\mathrm{d}t\right| \leqslant C \int_{t=(b_n \ln d)^2}^{+\infty} e^{-\frac{t}{4}} \,\mathrm{d}t \leqslant 4C \exp\left(-\frac{1}{4} b_n^2 (\ln d)^2\right) = O\!\left(d^{-1}\right). \tag{3.4.83}$$

By equations (3.4.82) and (3.4.83), we get:

$$\begin{aligned}\mathrm{Var}(|\mathrm{d}V_d|)\left(\phi_1, \phi_2\right) = d^{r-\frac{n}{2}} &\left(\int_M \phi_1 \phi_2 \left|\mathrm{d}V_M\right|\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \left(\frac{1}{2} \int_0^{+\infty} \left|D_{n,r}(t)\right| t^{\frac{n-2}{2}} \,\mathrm{d}t\right) \\ & + \left\|\phi_1\right\|_\infty \left\|\phi_2\right\|_\infty O\!\left(d^{r-\frac{n}{2}-\alpha}\right) + \left\|\phi_1\right\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\!\left(d^{r-\frac{n}{2}}\right). \quad (3.4.84)\end{aligned}$$

Finally, recall that we defined $I_{n,r}$ by eq. (3.1.6) and $D_{n,r}$ by Def. 3.4.24. Hence, we have:

$$\mathcal{I}_{n,r} = \frac{1}{2} \int_0^{+\infty} \left|D_{n,r}(t)\right| t^{\frac{n-2}{2}} \,\mathrm{d}t,$$

and this quantity is finite by Lemma 3.4.25. This concludes the proof of Theorem 3.1.6.

3.5 Proofs of the corollaries

3.5.1 Proof of Corollary 3.1.9

Corollary 3.1.9 is a direct consequence of Theorem 3.1.6 and the Markov inequality. Let $\phi \in \mathcal{C}^0(M)$, then, by (3.1.7) we have:

$$\mathrm{Var}(\langle |\mathrm{d}V_d|, \phi\rangle) = O\!\left(d^{r-\frac{n}{2}}\right),$$

where the error term depends on ϕ. Now, let $\alpha \geqslant \frac{r}{2} - \frac{n}{4}$ and $\varepsilon > 0$. We have:

$$\begin{aligned}\mathbb{P}\left(\left|\langle |\mathrm{d}V_d|, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d|, \phi\rangle]\right| > d^\alpha \varepsilon\right) &= \mathbb{P}\left(d^{-\alpha} \left|\langle |\mathrm{d}V_d|, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d|, \phi\rangle]\right| > \varepsilon\right) \\ &\leqslant \frac{1}{\varepsilon^2} \mathrm{Var}\!\left(d^{-\alpha} \langle |\mathrm{d}V_d|, \phi\rangle\right) \\ &\leqslant \frac{1}{\varepsilon^2} d^{-2\alpha} \mathrm{Var}(\langle |\mathrm{d}V_d|, \phi\rangle).\end{aligned}$$

3.5.2 Proof of Corollary 3.1.10

We obtain Cor. 3.1.10 as a consequence of Cor. 3.1.9. Let $U \subset M$ be an open subset. We denote by $\phi_U \in \mathcal{C}^0(M)$ the function such that $\phi_U(x)$ is the geodesic distance from x to the complement of U in (M, g). Then we have:

$$U = \{x \in M \mid \phi_U(x) > 0\},$$

and ϕ_U is non-negative. Hence, $Z_d \cap U = \emptyset$ if and only if $\langle |\mathrm{d}V_d| \,, \phi_U\rangle = 0$. Let $\varepsilon > 0$ such that:

$$\varepsilon < \frac{1}{2}\left(\int_M \phi_U \, |\mathrm{d}V_M|\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}.$$

Then, by Thm. 3.1.1, for d large enough we have:

$$d^{-\frac{r}{2}}\mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_U\rangle] - \varepsilon \geqslant \frac{1}{2}\left(\int_M \phi_U \, |\mathrm{d}V_M|\right) \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} > 0.$$

Thus, for d large enough, we have:

$$\begin{aligned}
\mathbb{P}\left(Z_d \cap U = \emptyset\right) &= \mathbb{P}\left(\langle |\mathrm{d}V_d| \,, \phi_U\rangle = 0\right) \\
&\leqslant \mathbb{P}\left(\langle |\mathrm{d}V_d| \,, \phi_U\rangle < \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_U\rangle] - d^{\frac{r}{2}}\varepsilon\right) \\
&\leqslant \mathbb{P}\left(\left|\langle |\mathrm{d}V_d| \,, \phi_U\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_U\rangle]\right| > d^{\frac{r}{2}}\varepsilon\right).
\end{aligned}$$

And by Cor. 3.1.9, this is a $O\left(d^{-\frac{n}{2}}\right)$.

3.5.3 Proof of Corollary 3.1.11

In this section we assume that $n \geqslant 3$. We consider a random sequence $(s_d)_{d\in\mathbb{N}}$ of sections of increasing degree, distributed according to the probability measure $\mathrm{d}\nu = \bigotimes_{d\in\mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Strictly speaking, $|\mathrm{d}V_{s_d}|$ is not defined for small d. However, $\mathrm{d}\nu$-almost surely, $|\mathrm{d}V_{s_d}|$ is well-defined for all $d \geqslant d_1$, so the statement of Cor. 3.1.11 makes sense.

Our proof follows the lines of the proof of Shiffman and Zelditch [SZ99, sect. 3.3] in the complex case. First, we prove that for every fixed $\phi \in \mathcal{C}^0(M)$ we have:

$$d^{-\frac{r}{2}} \langle |\mathrm{d}V_{s_d}| \,, \phi\rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\left(\int_M \phi \, |\mathrm{d}V_M|\right). \tag{3.5.1}$$

Then we use a separability argument to get the result. In the complex algebraic setting of [SZ99], the scaled volume of $s_d^{-1}(0) \subset \mathcal{X}$ is a deterministic constant, independent of d. In our real algebraic setting this is not the case.

Let $\phi \in \mathcal{C}^0(M)$, then we have:

$$\mathbb{E}\left[\sum_{d\in\mathbb{N}}\left(d^{-\frac{r}{2}}\left(\langle |\mathrm{d}V_{s_d}| \,, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi\rangle]\right)\right)^2\right] = \sum_{d\in\mathbb{N}} d^{-r}\,\mathrm{Var}(\langle |\mathrm{d}V_d| \,, \phi\rangle) < +\infty,$$

since $d^{-r}\,\mathrm{Var}(\langle |\mathrm{d}V_d| \,, \phi\rangle) = O\left(d^{-\frac{n}{2}}\right)$ by Cor. 3.1.7. Hence, $\mathrm{d}\nu$-almost surely, we have:

$$\sum_{d\in\mathbb{N}}\left(d^{-\frac{r}{2}}\left(\langle |\mathrm{d}V_{s_d}| \,, \phi\rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi\rangle]\right)\right)^2 < +\infty,$$

and
$$\left(d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi\right\rangle - d^{-\frac{r}{2}}\mathbb{E}[\langle|\mathrm{d}V_d|, \phi\rangle]\right) \xrightarrow[d\to+\infty]{} 0.$$
Then, by Thm. 3.1.1, $\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi\right\rangle$ satisfies (3.5.1) $\mathrm{d}\nu$-almost surely.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a dense sequence in the separable space $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$. Without loss of generality, we can assume that $\phi_0 = \mathbf{1}$, the unit constant function on M. Then, $\mathrm{d}\nu$-almost surely, we have:
$$\forall k \in \mathbb{N}, \qquad d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi_k\right\rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi_k \left|\mathrm{d}V_M\right|\right). \tag{3.5.2}$$

Let $\underline{s} = (s_d)_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ be a fixed sequence such that (3.5.2) holds. For every $\phi \in \mathcal{C}^0(M)$ and $k \in \mathbb{N}$ we have:
$$\begin{aligned}
&\left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi\right\rangle - \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi \left|\mathrm{d}V_M\right|\right)\right| \\
&\qquad\leqslant \left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi\right\rangle - d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi_k\right\rangle\right| \\
&\qquad\quad + \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left|\int_M \phi_k \left|\mathrm{d}V_M\right| - \int_M \phi \left|\mathrm{d}V_M\right|\right| \\
&\qquad\quad + \left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi_k\right\rangle - \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi_k \left|\mathrm{d}V_M\right|\right)\right| \\
&\qquad\leqslant \|\phi - \phi_k\|_\infty\left(d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \mathbf{1}\right\rangle + \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\mathrm{Vol}\left(M\right)\right) \\
&\qquad\quad + \left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi_k\right\rangle - \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi_k \left|\mathrm{d}V_M\right|\right)\right|.
\end{aligned}$$

Recall that $\phi_0 = \mathbf{1}$. Then, by (3.5.2), the sequence $(d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \mathbf{1}\right\rangle)_{d\in\mathbb{N}}$ converges. Hence it is bounded by some positive constant $K_{\underline{s}}$. Let $\phi \in \mathcal{C}^0(M)$ and let $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that:
$$\|\phi - \phi_k\|_\infty \leqslant \varepsilon\left(K_{\underline{s}} + \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\mathrm{Vol}\left(M\right)\right)^{-1}.$$
Then, for every d large enough we have:
$$\left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi_k\right\rangle - \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi_k \left|\mathrm{d}V_M\right|\right)\right| \leqslant \varepsilon,$$
and
$$\left|d^{-\frac{r}{2}}\left\langle \left|\mathrm{d}V_{s_d}\right|, \phi\right\rangle - \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\left(\int_M \phi \left|\mathrm{d}V_M\right|\right)\right| \leqslant 2\varepsilon.$$
Thus, ϕ satisfies (3.5.1).

Finally, whenever (3.5.2) is satisfied we have: for every $\phi \in \mathcal{C}^0(M)$, ϕ satisfies (3.5.1). Since the condition (3.5.2) is satisfied $\mathrm{d}\nu$-almost surely, this proves Cor. 3.1.11.

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Contributions à l'étude des sous-variétés aléatoires

Résumé. Dans cette thèse, nous étudions le volume et la caractéristique d'Euler de sous-variétés aléatoires de codimension $r \in \{1, \dots, n\}$ dans une variété ambiante M de dimension n. Dans un premier modèle, dit des ondes riemanniennes aléatoires, M est une variété riemannienne fermée. Nous considérons alors le lieu Z_λ des zéros communs de r combinaisons linéaires aléatoires indépendantes de fonctions propres du laplacien associées à des valeurs propres inférieures à $\lambda \geqslant 0$. Nous obtenons alors les asymptotiques du volume moyen et de la caractéristique d'Euler moyenne de Z_λ lorsque λ tend vers l'infini.

Dans un second modèle, M est le lieu réel d'une variété projective définie sur les réels. On s'intéresse dans ce cadre au lieu d'annulation réel Z_d d'une section holomorphe réelle globale aléatoire de $\mathcal{E} \otimes \mathcal{L}^d$, où $\mathcal{E}$ est un fibré hermitien de rang r, $\mathcal{L}$ est un fibré en droites hermitien ample et tous deux sont définis sur les réels. Nous estimons alors les moyennes du volume et de la caractéristique d'Euler de Z_d quand d tend vers l'infini. Dans ce modèle algébrique réel, nous calculons aussi l'asymptotique de la variance du volume de Z_d pour $1 \leqslant r < n$. Nous en déduisons, dans ce cas, des résultats asymptotiques d'équidistribution de Z_d dans M.

Mots-clés : volume, caractéristique d'Euler, sous-variétés aléatoires, ondes riemanniennes aléatoires, polynômes aléatoires, variété projective réelle, formule de Kac–Rice, noyau de Bergman.

Contributions to the study of random submanifolds

Abstract. We study the volume and Euler characteristic of codimension $r \in \{1, \dots, n\}$ random submanifolds in a dimension n manifold M. First, we consider Riemannian random waves. That is M is a closed Riemannian manifold and we study the common zero set Z_λ of r independent random linear combinations of eigenfunctions of the Laplacian associated to eigenvalues smaller than $\lambda \geqslant 0$. We compute estimates for the mean volume and Euler characteristic of Z_λ as λ goes to infinity.

We also consider a model of random real algebraic manifolds. In this setting, M is the real locus of a projective manifold defined over the reals. Then, we consider the real vanishing locus Z_d of a random real global holomorphic section of $\mathcal{E} \otimes \mathcal{L}^d$, where $\mathcal{E}$ is a rank r Hermitian vector bundle, $\mathcal{L}$ is an ample Hermitian line bundle and both these bundles are defined over the reals. We compute the asymptotics of the mean volume and Euler characteristic of Z_d as d goes to infinity. In this real algebraic setting, we also compute the asymptotic of the variance of the volume of Z_d, when $1 \leqslant r < n$. In this case, we prove asympotic equidistribution results for Z_d in M.

Keywords: volume, Euler characteristic, random submanifolds, Riemannian random waves, random polynomials, real projective manifold, Kac–Rice formula, Bergman kernel.

Image de couverture : réalisation d'une courbe aléatoire sur la sphère euclidienne de dimension 2 dans le modèle des ondes aléatoires monochromatiques avec $\lambda = 6480$; la courbe qui nous intéresse est la frontière entre les domaines blanc et noir.
Image réalisée par Alex Barnett (Dartmouth College) et reproduite avec sa permission.

ED 512 INFOMATHS Ecole doctorale

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