Curvatures

Exercise 1. Let (M, g) be a Riemannian manifold of dimension n.

1. If $n = 1$, what is its curvature?
2. If $n = 2$, how many degrees of freedom are there in the Riemann tensor? Give the expression in local coordinates of the Riemann, Ricci and scalar curvature of M.
3. How many degrees of freedom are there in the Riemann tensor for $n = 3$ and $n = 4$.

Exercise 2. Compute the Riemann, Ricci and scalar curvatures of the following Riemannian manifolds with their standard metric.

1. $\mathbb{R}^n$
2. $\mathbb{T}^n$
3. $\mathbb{S}^n$
4. $\mathbb{D}$ with $g_{\mathbb{D}} = \frac{4}{(1-\|x\|^2)^2} \langle \cdot , \cdot \rangle$.

Exercise 3.

1. Compute the sectional curvature of $\mathbb{S}^n_\rho$, the Euclidean sphere of radius ρ.
2. Compute the sectional curvature of $(\mathbb{D}, g_{\mathbb{D}})$.

Exercise 4 (Positive versus negative curvature)**.**

1. Let N denote the North pole of $\mathbb{S}^2$.
   (a) For $\rho \in (0, \pi)$, compute the volume of the geodesic ball $B(N, \rho)$. How does it compare to the volume of the ball of radius ρ in $\mathbb{R}^2$?
   (b) Compute the length of the circle $C(N, \rho)$. When ρ is small enough, how does it compare to its Euclidean analogue?
   (c) Let γ_1 and γ_2 be two geodesics on $\mathbb{S}^2$ such that $\gamma_i(0) = N$. We denote $v_i = \gamma_i'(0)$ and assume that $\|v_i\| = 1$. What is the distance between $\gamma_1(t)$ and $\gamma_2(t)$ for $t \in (-\pi, \pi)$?
   (d) When t is small, how does this distance compare to its Euclidean counterpart?
2. Same questions around 0 in the Poincaré disc $\mathbb{D}$, for ρ small.

Exercise 5. Let us consider normal coordinates $(x^1, \dots, x^n)$ around some point p in a Riemannian manifold (M, g). Let us denote as usual (g_{ij}) the matrix of g in these coordinates, (Γ^k_{ij}) the Christoffel symbols of the Levi–Civita connection and (R_{ijkl}) the components of the Riemann tensor (as a $\binom{4}{0}$-tensor). We admit that the following holds in these coordinates:

$$\forall i, j \in \{1, \dots, n\}, \qquad g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{1 \leqslant k,l \leqslant n} R_{iklj}(0) x^k x^l + O(\|x\|^3).$$

1. Give a two terms expansion of the Riemannian volume $\mathrm{d}V$ around 0 in these coordinates.
2. Give a two terms expansion of the volume of the geodesic ball of center p and radius ρ as $\rho \to 0$.