Tensor fields

Exercise 1. Let E and E' be two smooth vector bundles over the same base space M. Let $F : \Gamma(E) \to \Gamma(E')$ be a $\mathcal{C}^\infty(M)$-linear map.

1. Prove that there exists a unique $f : E \to E'$ which is a bundle map over M (that is, for any $x \in M$, $f_{|E_x} \in \mathrm{End}(E_x, E'_x)$) and satisfies:
$$\forall s \in \Gamma(E), \forall x \in M, \qquad F(s)(x) = f(s(x)). \tag{1}$$
2. Check that f can be seen as an element of $\Gamma(E^* \otimes E')$.
3. Conversely, check that a section $f \in \Gamma(E^* \otimes E')$ defines a unique $\mathcal{C}^\infty(M)$-linear map $F : \Gamma(E) \to \Gamma(E')$ satisfying (1).

Remark. In other terms we have defined a canonical isomorphism of $\mathcal{C}^\infty(M)$-modules between $\mathrm{Hom}(\Gamma(E), \Gamma(E'))$ and $\Gamma(E^* \otimes_{\mathbb{R}} E')$.

Similarly, we can show that $s \otimes s' \mapsto (x \mapsto s(x) \otimes s'(x))$ defines a canonical isomorphism of $\mathcal{C}^\infty(M)$-modules from $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E')$ to $\Gamma(E \otimes_{\mathbb{R}} E')$. For example, $\Gamma(\bigwedge^k T^*M \otimes_{\mathbb{R}} E)$ is isomorphic to $\Omega^k(M, E) := \Omega^k(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E)$, the space of k-forms with values in E.

Exercise 2. Let ∇ be a connection on a smooth vector bundle $E \to M$. We can see ∇ as a map from $\Omega^0(M, E)$ to $\Omega^1(M, E)$. We extend $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ by forcing the Leibniz rule:
$$\forall \alpha \in \Omega^k(M), \forall s \in \Gamma(E), \forall x \in M \qquad \nabla_x(\alpha \otimes s) = d_x\alpha \otimes s(x) + (-1)^k \alpha_x \wedge \nabla_x s$$
and $\mathbb{R}$-linearity.

1. Prove that $\nabla \circ \nabla$ is $\mathcal{C}^\infty(M)$-linear from $\Omega^0(M, E)$ to $\Omega^2(M, E)$.
2. Using the previous exercise, show that $\nabla \circ \nabla$ defines a section $R \in \Omega^2(M, \mathrm{End}(E))$. We call R the *curvature* of (E, ∇).
3. Let $(x_1, \dots, x_n)$ be local coordinates on M and $(e_1, \dots, e_r)$ be a local frame for E defined on the same open set U. We denote by (Γ_{ij}^k) the Christoffel symbols of ∇ in these coordinates. We also define (R_{ijk}^l) by:
$$\forall i, j, k \in \{1, \dots, n\}, \qquad R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) e_k = \sum_{l=1}^{n} R_{ijk}^l e_l.$$
Check that for any $i, j \in \{1, \dots, n\}$ and any $k, l \in \{1, \dots, r\}$ we have:
$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{m=1}^{r} \Gamma_{jk}^m \Gamma_{im}^l - \sum_{m=1}^{r} \Gamma_{ik}^m \Gamma_{jm}^l. \tag{2}$$
4. Check that for any $X, Y \in \Gamma(TM)$ and any $s \in \Gamma(E)$ we have:
$$\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s = R(X, Y)s.$$

Remark. In particular, if ∇ is the Levi–Civita connection of (M, g) then R is its Riemann curvature. Equation (2) is of course valid in this case.