# Geodesics

Let $(x_1, \dots, x_n)$ be local coordinates on an open subset of a Riemannian manifolds $(M, g)$, let $(g_{ij}(x))$ denote the matrix of $g$ in these coordinates and let $(g^{ij}(x))$ denote its inverse. Let $(\Gamma_{ij}^k(x))$ denote the Christoffel symbols associated with the Levi–Civita connection of $(M, g)$ in these coordinates. We recall that, for any $i, j$ and $k \in \{1, \dots, n\}$:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} \left( \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right).$$

**Exercise 1** (Image of a geodesic)**.** Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds and let $f : M_1 \to M_2$ be a smooth map.

1. If $f$ is an isometric diffeomorphism, is the image of a geodesic of $M_1$ a geodesic of $M_2$?
2. Same question if $f$ is a conformal diffeomorphism.
3. Same question if $f$ is an isometric embedding.

**Definition.** Let $(M, g)$ be a Riemannian manifold. We say that two maximal geodesics are *parallel* if they are either disjoint or equal up to reparametrization.

**Exercise 2** (Model spaces)**.**

1. Let us consider $\mathbb{R}^n$ with is canonical Euclidean metric.
   (a) What are the geodesics?
   (b) Compute the exponential map at any point $p \in \mathbb{R}^n$.
   (c) What is its injectivity radius?
   (d) Are there closed geodesics?
   (e) Are all geodesics closed?
   (f) Is the image of a geodesic a submanifold of the ambient space?
   (g) Let $\gamma$ be a geodesic and let $p \in \mathbb{R}^n \setminus \mathrm{Im}(\gamma)$. How many geodesics passing through $p$ and parallel to $\gamma$ are there?
2. Let $\alpha_1, \dots, \alpha_n > 0$, same questions for $\mathbb{T}_\alpha^n = \mathbb{R}^n / (\alpha_1 \mathbb{Z} \oplus \cdots \oplus \alpha_n \mathbb{Z})$ with the metric induced by the Euclidean one on $\mathbb{R}^n$.
3. Same questions on $\mathbb{S}^n$ with the metric induced by the Euclidean one on $\mathbb{R}^{n+1}$.
4. Same questions on the Poincaré disc $\mathbb{D}$ with the metric $g_{\mathbb{D}} := \frac{4}{(1-x^2-y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$
5. Same questions on the upper half-plane $\mathbb{H}$ with the metric $g_{\mathbb{H}} := \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$.

**Exercise 3** (Normal coordinates)**.** Let $(M, g)$ ba a Riemannian manifold and let $p \in M$.

1. Can we find local coordinates $(x_1, \dots, x_n)$ centered at $p$ such that the matrix $(g_{ij}(x))$ of $g$ in these coordinates satisfies $(g_{ij}(x)) = I_n + O(\|x\|)$, where $I_n$ is the identity matrix of size $n$? Don't forget to make sense of the $O(\|x\|)$.
2. Show that in the normal coordinates centered at $p$ we have: $(g_{ij}(x)) = I_n + O(\|x\|^2)$, and $\Gamma_{ij}^k(x) = O(\|x\|)$ for any $i, j$ and $k$.
3. Can we do better? $(I_n + O(\|x\|^3)$, constant equal to $I_n$, ...)