More connections

Exercise 1 (Derivatives at a vanishing point)**.** Let $E \to M$ be a vector bundle and let ∇ and $\widetilde{\nabla}$ be two connections on E. Let $s \in \Gamma(E)$ and $x \in M$ be such that $s(x) = 0$. Compare $\nabla_x s$ and $\widetilde{\nabla}_x s$.

Exercise 2. Let $E \to M$ be a vector bundle eqquiped with a connection ∇. Let $x_0 \in M$ and $y \in E_{x_0}$, is there a local (resp. global) section $s \in \Gamma(E)$ such that $s(x_0) = y$ and $\nabla_{x_0} s = 0$?

Exercise 3. Let s be a smooth section of the vector bundle $E \to M$ and let $x_0 \in M$. Is there a connection ∇ on E such that ∇s vanishes on some neighborhood of x_0 in M?

Exercise 4 (Christoffel symbols)**.**

1. Let (M, g) be a Riemannian manifold. We denote by $(x_1, \ldots, x_n)$ local coordinates on a open subet U of M and by $G = (g_{ij})$ the matrix of g in these coordinates. Let (g^{kl}) denote the coefficients of G^{-1}. Check that the Christoffel symbols $(\Gamma_{ij}^k)_{1 \leqslant i,j,k \leqslant n}$ of the Levi–Civita connection ∇ of (M, g) are symmetric in (i, j). Prove that for any $i, j, k \in \{1, \ldots, n\}$ we have:
$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right).$$
2. Recall that the half-plane model of the hyperbolic plane is $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ endowed with the metric $g_{(x,y)} := \frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. Compute the covariant derivatives of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ for the Levi–Civita connection.
3. Recall that the Poincaré disc is the unit open disc $\mathbb{D}^2 \subset \mathbb{R}^2$ endowed with the metric $g_{(x,y)} := \frac{4}{(1-x^2-y^2)^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. Compute the covariant derivatives for the Levi–Civita connection of the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ associated with the polar coordinates on $\mathbb{D}^2 \setminus \{0\}$.