# Connections

**Exercise 1** (Warm-up)**.** Let $g_0$ denote the Euclidean metric of $\mathbb{R}$, we define another Riemannian metric $g$ on $\mathbb{R}$ by $g_x := e^{-x^2}(g_0)_x$ for any $x \in \mathbb{R}$. Let $\nabla_0$ (resp. $\nabla$) denote the Levi–Civita connection on $T\mathbb{R}$ associated with $g_0$ (resp. $g$). Compute $(\nabla_0)\frac{\partial}{\partial x}$ and $\nabla\frac{\partial}{\partial x}$.

**Exercise 2** (Dual connection)**.** Let $\nabla$ be a connection on a vector bundle $E \to M$. We still denote by $\nabla$ the induces connections on bundles of the form $E \otimes \cdots \otimes E \otimes E^* \otimes \cdots \otimes E^* \to M$.

1. Compute $d(\alpha(X))$ where $\alpha \in \Gamma(E^*)$ and $X \in \Gamma(E)$.
2. Compute $\nabla \operatorname{Id}$ where $\operatorname{Id} : x \mapsto \operatorname{Id}_{E_x}$ is a section of $\operatorname{End}(E) \to M$.

**Exercise 3** (Hessian and torsion)**.** Let $M$ be a manifold and let $f : M \to \mathbb{R}$ be smooth.

1. In a chart $(U, \varphi)$, we can compute the second diffenrential of $f_\varphi := f \circ \varphi^{-1}$. How do chart transitions act on $D^2 f_\varphi$? Can we define an intrinsic notion of second differential of $f$ that would read as $D^2 f_\varphi$ in the chart $(U, \varphi)$, for any such chart?
2. Let us now assume that $TM$ is endowed with a connection $\nabla$. Let $\nabla^2 f$ be the section of $T^*M \otimes T^*M$ defined by:

   $$\forall x \in M,\ \forall u, v \in T_x M, \quad \nabla^2_x f(u, v) := (\nabla_u(df)) \cdot v.$$

   Let $X$ and $Y$ be two vectors fields on $M$, prove that $\nabla^2 f(X, Y) = X \cdot (Y \cdot f) - (\nabla_X Y) \cdot f$.
3. Give an necessary and sufficient condition on $\nabla$ ensuring that, for all $f \in \mathcal{C}^\infty(M)$, for all $x \in M$, $\nabla^2_x f$ is symmetric.