Riemannian metrics, isometries

Exercise 1 (Standard metrics)**.** Let g_0 denote the standard Riemannian metric on $\mathbb{R}^n$, that is for all $x \in \mathbb{R}^n$, $(g_0)_x$ is the standard Euclidean inner product.

1. Are the translations isometries of $(\mathbb{R}^n, g_0)$? Define a natural Riemannian metric on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

2. Let $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ denote the canonical inclusion, we set $g = i^*(g_0)$. Do elements of the orthogonal group $O_{n+1}(\mathbb{R})$ induce isometries of $(\mathbb{S}^n, g)$? Define a natural Riemannian metric on the real projective space $\mathbb{RP}^n$.

Exercise 2 (Dimension 1)**.** 1. Let M be a connected smooth manifold without boundary of dimension 1. Are any two Riemannian metrics on M conformal to one another?

2. Same question if $\dim(M) \geqslant 2$.

3. What are the connected Riemannian manifolds without boundary of dimension 1 up to isometries?

Exercise 3 (Hyperbolic spaces)**.** Let $B = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$ denote the standard Lorentz form on $\mathbb{R}^{n+1}$. Let $\mathcal{H}^n$ denote the set $\{x \in \mathbb{R}^{n+1} \mid B(x,x) = -1, x_0 > 0\}$. We also denote by $\mathbb{H}^n$ the half-space $\{x \in \mathbb{R}^n \mid x_n > 0\}$ and by $\mathbb{D}^n$ the unit open ball in $\mathbb{R}^n$. Prove that the following are Riemannian manifolds that are isometric to one another:

- $\mathcal{H}^n$ endowed with the restriction of B,
- $\mathbb{H}^n$ with the metric $\frac{1}{|x_n|^2} \sum_{i=1}^n dx^i \otimes dx^i$,
- $\mathbb{D}^n$ with the metric $\frac{4}{(1-\|x\|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i$

Exercise 4 (Hyperbolic half-plane and Poincaré disc)**.** In this exercise, we simply denote by $\mathbb{H}$ the hyperbolic half-plane $\mathbb{H}^2$ and by $\mathbb{D}$ the hyperbolic disc $\mathbb{D}^2$ (also called *Poincaré disc*).

1. Check that conformal diffeomorphisms of $\mathbb{D}$ (resp. $\mathbb{H}$) preserving the orientation are biholomorphisms.

2. Describe the conformal diffeomorphisms of $\mathbb{D}$ (resp. $\mathbb{H}$). Which one are isometries?