Vector bundles

Exercise 1. Let M be a smooth manifold. What are the global sections of $M \times \mathbb{R}^k \to M$, $TM \to M$ and $\bigwedge^k T^*M \to M$?

Exercise 2 (Frames)**.** Let $E \to M$ be a smooth vector bundle of rank k. A *local frame* for E over the open subset $U \subset M$ is a family $(s_1, \ldots, s_k)$ of smooth sections of $E_{|U} \to U$ such that, for any $x \in U$, $(s_1(x), \ldots, s_k(x))$ is a basis of the fiber E_x.

1. Check that it is equivalent to give a local frame for E over U or a local trivialization $E_{|U} \simeq U \times \mathbb{R}^k$.
2. Give a necessary and sufficient condition on global sections of $E \to M$ for this bundle to be trivial.
3. Is $T\mathbb{T}^n \to \mathbb{T}^n$ trivial? Is $T\mathbb{S}^2 \to \mathbb{S}^2$ trivial?
4. Does any smooth vector bundle admit a non-zero smooth section? A non-vanishing smooth section?

Exercise 3 (Pullback)**.**
1. Is the pullback of a trivial vector bundle trivial?
2. Let $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ be the canonical projection, compare $T\mathbb{S}^n \to \mathbb{S}^n$ and $\pi^*(T\mathbb{RP}^n) \to \mathbb{S}^n$.

Exercise 4 (Sub-bundles)**.** Let $E \to M$ be a smooth vector bundle of rank r and let $V \subset E$. We say that $V \to M$ is a *sub-bundle* of $E \to M$ of rank k if, for any $x \in M$, there exists a local trivialization $\phi : E_{|U} \simeq U \times \mathbb{R}^r$ of E over a neighborhood U of x such that:

$$\phi\left(E_{|U} \cap V\right) = U \times \left(\mathbb{R}^k \times \{0\}\right).$$

1. Check that a sub-bundle of a smooth vector bundle is a smooth vector bundle.
2. Let N be a smooth submanifold of M and let $i : N \to M$ be the inclusion. Is TN a sub-bundle of $i^*(TM)$?

Exercise 5 (Group of line bundles)**.**
1. Let $L \to M$ be a line bundle on a smooth manifold, is $L \otimes L^* \to M$ trivial?
2. Define an abelian group structure on the set of line bundles over M up to isomorphism.

Exercise 6 (Tautological line bundle)**.** Let $J = \left\{(D, x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \;\middle|\; x \in D\right\}$, we say that $J \to \mathbb{RP}^n$ is the *tautological line bundle* over $\mathbb{RP}^n$.

1. Check that J is a sub-bundle of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ of rank 1. Is it trivial?
2. Let $L = J^*$, check that an homogeneous polynomial P of degree d in $(n+1)$ variables defines a global section of $L^{\otimes d} \to \mathbb{RP}^n$.
3. For which $d \in \mathbb{Z}$ is $L^{\otimes d} \to \mathbb{RP}^n$ trivial?

Exercise 7 (Line bundles on the sphere)**.**
1. What are the line bundles over $\mathbb{S}^1$ up to isomorphism? Which one is $T\mathbb{S}^1$?
2. Describe the group structure defined in exercise 5 when $M = \mathbb{S}^1$.
3. What are the line bundles over $\mathbb{S}^n$ up to isomorphism when $n \geqslant 2$?