Geometric meaning of connections

Definition. A *metric* on a vector bundle $E \to M$ is a section $h \in \Gamma(E^* \otimes E^*)$ such that, for all $x \in M$, h_x is an inner product on E_x.

Definition. A connection ∇ on a vector bundle $E \to M$ eqquiped with a metric h is said to be *compatible* with h if, for any $s_1, s_2 \in \Gamma(E)$, $dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$.

In the sequel, let $p : E \to M$ be a rank r vector bundle and let n denote the dimension of M.

Definition. For any $y \in E$, we denote $V_y = \ker d_y p$. Then, $V \to E$ is a rank r sub-bundle of $TE \to E$ and is called the *vertical sub-bundle* of TE.

Definition. An *horizontal sub-bundle* of TE is a sub-bundle $H \to E$ of $TE \to E$ such that, for any $y \in E$, $H_y \otimes V_y = T_yE$.

Remarks.

- Note that H has rank n and that for any $y \in E$, $V_y = T_y(E_x)$ is canonically isomorphic to E_x, where $x = p(y)$.
- In the literature, horizontal sub-bundles are called *Ehresmann connections*, we don't use this terminology in order to avoid confusing horizontal sub-bundles with connections (in the sense of the course).

For any $\lambda \in \mathbb{R}$, let $M_\lambda : E \to E$ denote the fiberwise multiplication by λ.

Let $\Delta : M \to M \times M$ be defined by $\Delta(x) = (x, x)$. Then $\Delta^*(E \times E) \to M$ is the space

$$\left\{(x, y, y') \in M \times E \times E \;\middle|\; p(y) = x = p(y')\right\} \simeq \left\{(y, y') \in E \times E \;\middle|\; p(y) = p(y')\right\}$$

with the natural projection. We denote by $A : \Delta^*(E \times E) \to E$ the fiberwise addition of E.

Definition. We say that an horizontal sub-bundle H of TE is *linear* if:

- for any $y, y' \in E$ such that $p(y) = p(y')$ we have:
$$d_{(y,y')}A\left((H_y \times H_{y'}) \cap T_{(y,y')}\Delta^*(E \times E)\right) = H_{A(y,y')};$$
- for any $\lambda \in \mathbb{R}$ and $y \in E$ we have $d_yM_\lambda(H_y) = H_{M_\lambda(y)}$.

The main goal of the following exercise is to prove that the choice of a linear horizontal sub-bundle of TE is equivalent to the choice of a connection on E.

Exercise 1. Let $p : E \to M$ be a vector bundle of rank r on a n-dimensional basis.

1. Let $H \to E$ be a linear horizontal sub-bundle of TE. Define a connection ∇ on E associated with H.
   Hint: consider the projection onto V_y along H_y.
2. Conversely, assume that E is eqquiped with a connection ∇. Define a linear horizontal sub-bundle H of TE associated with ∇.
   Hint: consider the image of d_xs, where $s \in \Gamma(E)$ is such that $\nabla_x s = 0$.
3. Recall that if $s(x) = 0$ then $\nabla_x s$ does not depend on ∇. What does it mean in terms of the associated linear horizontal sub-bundles?
4. Let h be a metric on E and let ∇ be a connection on E compatible with h. What does it mean for the associated linear horizontal sub-bundle?