Geometric meaning of connections (Solution)

1. Let $x \in M$ and $y \in E_x$, we denote by Π_y the projection onto V_y along H_y in T_yE. For any $s \in \Gamma(E)$, we define $\forall x \in M$, $\nabla_x s := \Pi_{s(x)}(d_x s)$. We have $\nabla_x s : T_xM \to V_y \simeq E_x$, hence $\nabla s \in \Gamma(T^*M \otimes E)$ and $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$. Let us check that ∇ is a connection.

Homogeneity. Let $\lambda \in \mathbb{R}$ and $s \in \Gamma(E)$. Let $x \in M$, we denote $y = s(x)$, we have $d_x s = \nabla_x s + (d_x s - \nabla_x s)$ with $\nabla_x s$ taking values in V_y and $(d_x s - \nabla_x s)$ taking values in H_y. We have

$$d_x(\lambda s) = d_x(M_\lambda \circ s) = d_y M_\lambda \circ d_x s = d_y M_\lambda(\nabla_x s) + d_y M_\lambda(d_x s - \nabla_x s).$$

We assumed that $d_y M_\lambda(H_y) = H_{\lambda y}$, hence $d_y M_\lambda(d_x s - \nabla_x s)$ takes values in $H_{\lambda y}$. Besides, $p \circ M_\lambda = p$, hence $d_{\lambda y} p \circ d_y M_\lambda = d_y p$, and $d_y M_\lambda(V_y) = d_y M_\lambda(\ker d_y p) \subset \ker d_{\lambda y} p = V_{\lambda y}$. In fact $d_{\lambda y} p \circ d_y M_\lambda = V_{\lambda y}$ by a dimension argument. Thus $d_y M_\lambda(\nabla_x s)$ takes values in $V_{\lambda y}$ and we have $\nabla_x(\lambda s) = d_y M_\lambda(\nabla_x s)$.

Recall that $V_y \simeq E_x \simeq V_{\lambda y}$ canonically, and under this identification $d_y M_\lambda$ is the multiplication by λ in E_x. Thus $\nabla_x(\lambda s) = \lambda \nabla_x s$.

Additivity. Now, let $s_1, s_2 \in \Gamma(E)$, let $x \in M$, we denote $y_1 = s_1(x)$ and $y_2 = s_2(x)$. We have:

$$d_x(s_1 + s_2) = d_x(A \circ (s_1, s_2)) = d_{(y_1,y_2)} A \circ d_x(s_1, s_2)$$

and

$$d_x(s_1, s_2) = (d_x s_1, d_x s_2) = (\nabla_x s_1, \nabla_x s_2) + (d_x s_1 - \nabla_x s_1, d_x s_2 - \nabla_x s_2).$$

On the one hand, $(\nabla_x s_1, \nabla_x s_2)$ takes values in $(V_{y_1} \times V_{y_2}) \subset T_{(y_1,y_2)}\Delta^*(E \times E)$. Note that:

$$T_{(y_1,y_2)}\Delta^*(E \times E) = \{(v_1, v_2) \in T_{y_1}E \times T_{y_2}E \mid d_{y_1}p \cdot v_1 = d_{y_2}p \cdot v_2\}.$$

Since, $\Delta \circ p \circ A = (p, p) : \Delta^*(E \times E) \to M \times M$ and Δ is an immersion, we have $d_{(y_1,y_2)}A(\ker(d_{y_1}p, d_{y_2}p)) \subset \ker d_{A(y_1,y_2)}p$. Thus $d_{(y_1,y_2)}A(V_{y_1} \times V_{y_2})$ is a subspace of $V_{y_1+y_2}$ and $d_{(y_1,y_2)}A \circ (\nabla_x s_1, \nabla_x s_2)$ takes values in $V_{y_1+y_2}$.

On the other hand, $(d_x s_1 - \nabla_x s_1, d_x s_2 - \nabla_x s_2)$ takes values in $H_{y_1} + H_{y_2}$. Moreover,

$$d_{y_i}p \circ (d_x s_i - \nabla_x s_i) = d_{y_i}p \circ d_x s_i = d_x(p \circ s_i) = \mathrm{Id},$$

which means that the image of $(d_x s_1 - \nabla_x s_1, d_x s_2 - \nabla_x s_2)$ is in $T_{(y_1,y_2)}\Delta^*(E \times E)$. Since H is linear, the image of $d_{(y_1,y_2)}A \circ (d_x s_1 - \nabla_x s_1, d_x s_2 - \nabla_x s_2)$ is included in $H_{y_1+y_2}$.

Finally, we get that:

$$\begin{aligned}\nabla_x(s_1 + s_2) &= \Pi_{y_1+y_2} \circ \left(d_{(y_1,y_2)}A \circ (\nabla_x s_1, \nabla_x s_2) + d_{(y_1,y_2)}A \circ (d_x s_1 - \nabla_x s_1, d_x s_2 - \nabla_x s_2)\right) \\ &= d_{(y_1,y_2)}A \circ (\nabla_x s_1, \nabla_x s_2).\end{aligned}$$

Under the canonical identifications $V_{y_1+y_2} \simeq E_x$ and $V_{y_1} \simeq E_x \simeq V_{y_2}$, $d_{(y_1,y_2)}A$ reads as the addition of E_x. Hence, $\nabla_x(s_1 + s_2) = \nabla_x s_1 + \nabla_x s_2$ and ∇ is linear.

Leibniz's rule. Let $s \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$. Let $x \in M$, we denote $y = s(x)$ and $\lambda = f(x)$. Let $\varphi_{|U} : E_{|U} \to U \times \mathbb{R}^r$ be a local trivialization of E on a neighborhood U of x. We have $\varphi_{|U} \circ s = (\mathrm{Id}_U, \sigma)$ for some smooth $\sigma : U \to \mathbb{R}^r$. Then, $\varphi_{|U} \circ (fs) = (\mathrm{Id}_U, f\sigma)$ and:

$$\begin{aligned} d_{\lambda y}\varphi_{|U} \circ d_x(fs) = d_x(\mathrm{Id}_U, f\sigma) &= (\mathrm{Id}_{T_xU}, d_xf \otimes \sigma(x) + f(x)d_x\sigma) \\ &= (\mathrm{Id}_{T_xU}, \lambda d_x\sigma) + d_xf \otimes (0, \sigma(x)). \end{aligned}$$

We have $\varphi_{|U} \circ M_\lambda \circ s = (\mathrm{Id}_U, \lambda s)$, hence $d_{\lambda y}\varphi_{|U} \circ d_yM_\lambda \circ d_xs = (\mathrm{Id}_{T_xU}, \lambda d_x\sigma)$. Moreover, if we see $s(x) \in E_x$ as an element of $V_{\lambda y} \subset T_{\lambda y}E$, we have $d_{\lambda y}\varphi_{|U} \cdot s(x) = (0, \sigma(x))$. Finally, we get: $d_{\lambda y}\varphi_{|U} \circ d_x(fs) = d_{\lambda y}\varphi_{|U} \circ d_yM_\lambda \circ d_xs + d_{\lambda y}\varphi_{|U}(d_xf \otimes s(x))$, and

$$d_x(fs) = d_yM_\lambda \circ d_xs + d_xf \otimes s(x). \tag{1}$$

We have already seen that $d_yM_\lambda \circ d_xs = d_yM_\lambda(\nabla_xs) + d_yM_\lambda(d_xs - \nabla_xs)$ where the first term takes values in $V_{\lambda y}$ and the second one takes values in $H_{\lambda y}$. Since $s(x) \in E_x \simeq V_{\lambda y}$, we get$\nabla_x(fs) = d_yM_\lambda(\nabla_xs) + d_xf \otimes s(x)$. Using once again that $V_y \simeq E_x \simeq V_{\lambda y}$ and the fact that d_yM_λ reads as the multiplication by λ of E_x under these identifications, we proved that $\nabla_x(fs) = f(x)\nabla_xs + d_xf \otimes s(x)$.

Conclusion. ∇ defined by $\nabla_xs = \Pi_{s(x)} \circ d_xs$ is a $\mathbb{R}$-linear map $\Gamma(E) \to \Gamma(T^*M \otimes E)$ that satisfies Leibniz's rule. Hence it is a connection on E.

2. Let $s \in \Gamma(E)$, $x \in M$ and $y = s(x)$. Since $d_yp \circ d_xs = \mathrm{Id}_{T_xM}$, d_xs is injective from T_xM to T_yE and its image is transverse to $\ker d_yp = V_y$. Thus $d_xs(T_xM)$ is an horizontal direction in E_y ... that depends heavily on s. The point is to prove that the image of d_xs is the same for all $s \in \Gamma(E)$ such that $s(x) = y$ and $\nabla_xs = 0$. Then we can define H_y as the image of d_xs for any such section.

Sections with vanishing derivative. Let $(e_1, \dots, e_r)$ be a local frame defined on a neighborhood U of x and let $(x^1, \dots, x^n)$ be local coordinates on U centered at x. We denote by (Γ_{ij}^k) the Christolffel symbols of ∇ associated with the frame (e_i) and these coordinates. Let $s = \sum_{i=1}^r f^ie_i$ be a smooth section of $E_{|U}$. Then, ∇_xs equals:

$$\begin{aligned} \sum_{i=1}^r d_xf^i \otimes e_i(x) + f^i(x)\nabla e_i(x) &= \sum_{i=1}^r d_xf^i \otimes e_i(x) + f^i(x)\sum_{j=1}^n\sum_{k=1}^r \Gamma_{ji}^k(x)\,\mathrm{d}x^j \otimes e_k(x) \\ &= \sum_{i=1}^r\sum_{j=1}^n \left(\frac{\partial f^i}{\partial x_j}(x) + \sum_{k=1}^r \Gamma_{jk}^i(x)f^k(x)\right)\mathrm{d}x^j \otimes e_i(x). \end{aligned}$$

Thus $s(x) = y = \sum y^ie_i(x)$ and $\nabla_xs = 0$ if and only if:

$$\begin{cases} \forall i \in \{1, \dots, r\}, & f^i(x) = y^i \\ \forall i \in \{1, \dots, r\}, \forall j \in \{1, \dots, n\}, & \dfrac{\partial f^i}{\partial x_j}(x) = -\displaystyle\sum_{k=1}^r \Gamma_{jk}^i(x)y^k. \end{cases} \tag{2}$$

First, this proves that there exists s such that $s(x) = y$ and $\nabla_xs = 0$. We define such a section locally using the frame (e_i) and the coordinates $(x^1, \dots, x^n)$ by $s = \sum f^ie_i$ with:

$$\forall i \in \{1, \dots, r\}, \qquad f^i(x_1, \dots, x_n) = y^i - \sum_{j=1}^n x^j\left(\sum_{k=1}^r \Gamma_{jk}^i(0)y^k\right).$$

Then we extend s into a global section of E using a smooth bump function that equals 1 in a neighborhood of x and whose support in contained in U.

Let $s \in \Gamma(E)$ be such that $s(x) = y$ and $\nabla_x s = 0$. We write $s = \sum f^i e_i$ in a local chart. Using Eq. (1), which is still valid in this context, we get:

$$d_x s = \sum d_x(f^i e_i) = \sum d_{e_i(x)} M_{f^i(x)} \circ d_x e_i + d_x f^i \otimes e_i(x).$$

Then, by Eq. (2), in local coordinates we get:

$$d_x s = \sum_{i=1}^{r} d_x(y^i e_i) - \sum_{j=1}^{r} \sum_{k=1}^{r} \Gamma^i_{jk}(x) y^k \, \mathrm{d}x^j \otimes e_i(x). \tag{3}$$

Note that the right-hand side no longer depends on $f = (f^1, \ldots, f^n)$. It only depends on ∇ and our choice of coordinates. Thus all sections $s \in \Gamma(E)$ such that $s(x) = y$ and $\nabla_x s = 0$ have the same differential. We define $H_y := d_x s(T_x M)$ for any such section. We have already seen that for any $y \in E$, H_y is transverse to V_y and that $d_x s$ is injective. Then $\dim(H_y) = \dim(M) = n$, and since $\dim(V_y) = r$ we have $H_y \oplus V_y = T_y E$.

Horizontal sub-bundle Let $y \in E$, we denote $x = p(y)$. Let $(e_1, \ldots, e_r)$ be a local frame around x and let $(x^1, \ldots, x^n)$ be local coordinates defined on the same neighborhood U of x. From the definition of H_y, we see that $(d_x s \cdot \frac{\partial}{\partial x_1}, \ldots, d_x s \cdot \frac{\partial}{\partial x_n})$ is a basis of H_y, where $d_x s : T_x M \to T_y E$ is defined by Eq. (3). Note that we don't use the fact that it is the differential of something, the notation $d_x s$ is formal here.

For any $j \in \{1, \ldots, n\}$, we have: $d_x s \cdot \dfrac{\partial}{\partial x_j} = \displaystyle\sum_{i=1}^{r} d_x(y^i e_i) \cdot \dfrac{\partial}{\partial x_j} - \sum_{k=1}^{r} \Gamma^i_{jk}(x) y^k e_i(x).$

We define smooth local vector fields $X_1, \ldots, X_n$ on $E_{|U}$ by:

$$X_j : y \longmapsto \sum_{i=1}^{r} d_{p(y)}(y^i e_i) \cdot \frac{\partial}{\partial x_j} - \sum_{k=1}^{r} \Gamma^i_{jk}(p(y)) y^k e_i(p(y)).$$

Then, for any $y \in E_{|U}$, $(X_1(y), \ldots, X_n(y))$ is a basis of H_y. For $j \in \{1, \ldots, r\}$, we define $X_{n+j} : y \mapsto e_j(p(y)) \in E_{p(y)} \simeq V_y \subset T_y E$. Then $X_{n+1}, \ldots, X_{n+r}$ are smooth vector fields on $E_{|U}$ such that $(X_{n+1}(y), \ldots, X_{n+r}(y))$ is a basis of V_y for any $y \in E_{|U}$. Thus $(X_1, \ldots, X_{n+r})$ is a local frame for TE on $E_{|U}$ such that $\forall y \in E_{|U}$, $(X_1(y), \ldots, X_n(y))$ is a basis of H_y. This proves that $H \to E$ is an horizontal sub-bundle of $TE \to E$.

Linearity of H We now need to check that H is linear. Let $y \in E$, $x = p(y)$ and $\lambda \in \mathbb{R}$. There exists $s \in \Gamma(E)$ such that $s(x) = y$ and $\nabla_x s = 0$. Then $M_\lambda \circ s(x) = M_\lambda(y)$ and $\nabla_x(M_\lambda \circ s) = \nabla_x(\lambda s) = \lambda \nabla_x s = 0$, the operator ∇ being $\mathbb{R}$-linear. By definition $H_{M_\lambda(y)} = d_x(M_\lambda \circ s)(T_x M) = (d_y M_\lambda \circ d_x s)(T_x M) = d_y M_\lambda(H_y)$.

Similarly, let $y_1, y_2 \in E$ such that $p(y_1) = p(y_2) = x \in M$. For $i \in \{1, 2\}$, let $s_i \in \Gamma(E)$ such that $s_i(x) = y_i$ and $\nabla_x s_i = 0$, so that $H_{y_i} = d_x s_i(T_x M)$. Then $A \circ (s_1, s_2) \in \Gamma(E)$ is such that $(A \circ (s_1, s_2))(x) = y_1 + y_2$ and $\nabla_x(A \circ (s_1, s_2)) = \nabla_x(s_1 + s_2) = \nabla_x s_1 + \nabla_x s_2 = 0$. Thus $H_{y_1+y_2} = d_x(A \circ (s_1, s_2))(T_x M) = d_{(y_1,y_2)} A \circ d_x(s_1, s_2)(T_x M)$. One can check that:

$$d_x(s_1, s_2)(T_x M) = (d_x s_1, d_x s_2)(T_x M) = (H_{y_1} \times H_{y_2}) \cap T_{(y_1,y_2)} \Delta^*(E \times E).$$

This shows that H is compatible with A and concludes the proof of the linearity of H.

3. Recall that the zero section of E is $z : M \to E$ defined by $z(x) = 0 \in E_x$. Since z and p are smooth and $p \circ z = \mathrm{Id}_M$, z is an embedding of M into E. Indeed, z is an immersive injection and is proper. Let us denote $Z = z(M)$ the image of the zero section.

   Let $y \in Z$ and $x = p(y)$. Then $y = z(x)$ and $T_yE = T_yZ \oplus V_y$. Indeed, $d_yp \circ d_xz = \mathrm{Id}_{T_xM}$ so that $T_yZ = d_xz(T_xM)$ has dimension n and is transverse to $V_y = \ker(d_yp)$. That is, we already have a canonical horizontal direction in T_yE which is T_yZ.

   Let ∇ be any connection on E and let H be the associated linear horizontal sub-bundle of TE. Let $f : M \to \mathbb{R}$ be constant equal to 0. Let also $x \in M$ and $y = z(x) \in E$. We have: $\nabla_x z = \nabla_x(fz) = d_xf \otimes z(x) + f(x)\nabla_x z = 0$. Then $z \in \Gamma(E)$ is such that $z(x) = y$ and $\nabla_x z = 0$. Thus, by the previous question, $H_y = d_xz(T_xM) = T_yZ$.

   In conclusion, let $s \in \Gamma(E)$ and $x \in M$ be such that $y = s(x) = 0$. For any connection ∇, the associated horizontal direction in T_yE is $H_y = T_yZ$. Since $\nabla_x s$ is the projection of d_xs onto V_y along H_y, it does not depend on the choice of ∇.

4. We defined $h \in \Gamma(E^* \otimes E^*)$. We can also see h as a smooth map from $\Delta^*(E \times E)$ to $\mathbb{R}$, where $\Delta : M \to M \times M$ is defined by $x \mapsto (x, x)$. We define similarly $\Delta_E : E \to E \times E$ by $\Delta_E(y) = (y, y)$. For any $R > 0$, we denote by $\mathcal{T}_R$ the tube of radius R in E:

$$\mathcal{T}_R := \left\{ y \in E \;\middle|\; h_{p(y)}(y, y) = R^2 \right\} = (h \circ \Delta_E)^{-1}(R^2).$$

   We will prove that $\mathcal{T}_R$ is a smooth hypersurface of E for any $R > 0$ and that, if ∇ is a metric connection on (E, h) and H is the associated linear horizontal sub-bundle of TE, then for every $y \in \mathcal{T}_R$, H_y is tangent to $\mathcal{T}_R$ at y.

   Note that we say nothing about what happens along Z (that we can think of as $\mathcal{T}_0$) but, by the previous question, H_y does not depend on ∇ if $y \in Z$. In particular, it does not depend on the fact that ∇ be compatible with h.

   Tubes. First note that $h \circ \Delta_E : E \to \mathbb{R}_+$ is smooth. Let $R > 0$ and let $y \in \mathcal{T}_R$. For any $t > 0$, we have $h \circ \Delta_E(ty) = h(ty, ty) = t^2 h(y, y) = t^2 R^2$. Taking the derivative of this expression at $t = 1$ we get: $d_y(h \circ \Delta_E) \cdot y = 2R^2$ (recall that $y \in E_x \simeq V_y \subset T_yE$). Hence $d_y(h \circ \Delta_E) \neq 0$. Thus $h \circ \Delta_E$ is a submersion on $E \setminus Z$ and, for any $R > 0$, $\mathcal{T}_R$ is smooth hypersurface of E.

   Tangency. Let $R > 0$ and $y \in \mathcal{T}_R$, we denote $x = p(y)$. Let ∇ be a connection on E that is compatible with h and let H denote the associated horizontal sub-bundle of TE. Let $s \in \Gamma(E)$ such that $s(x) = y$ and $\nabla_x s = 0$, so that $H_y = d_xs(T_xM)$. We have:

$$d_y(h \circ \Delta_E) \circ d_xs = d_x(h \circ \Delta_E \circ s) = d_x(h(s, s)) = 2h_y(\nabla_x s, s(x)) = 0,$$

   where we used the compatibility of ∇ with h, the symmetry of h, and $\nabla_x s = 0$. Finally,

$$H_y = d_xs(T_xM) \subset \ker d_y(h \circ \Delta_E) = T_y\mathcal{T}_R.$$

   That is, the horizontal sub-bundle $H \to E$ associated with a connection compatible with the metric h on E is everywhere tangent to the tubes of constant radius in (E, h). Note that this is also true for $R = 0$ since $\mathcal{T}_0 = Z$ and, for any $y \in Z$, $H_y = T_yZ$.