Memo curvatures

Let (M, g) be a Riemannian manifold and ∇ denote its Levi–Civita connection.

1 Definitions

Riemann curvature. The *Riemann curvature* of (M, g) is the $\binom{3}{1}$-tensor R defined by:

$$\forall X, Y, Z \in \Gamma(TM), \qquad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1}$$

We denote by $\widetilde{R}$ the *fully-covariant* version of the Riemann tensor, that is the $\binom{4}{0}$-tensor defined by:

$$\forall X, Y, Z, T \in \Gamma(TM), \qquad \widetilde{R}(X, Y, Z, T) = g(R(X, Y)Z, T). \tag{2}$$

On other terms, $\widetilde{R}(X, Y, Z, \cdot) = (R(X, Y)Z)^\flat$, where $^\flat : T_xM \to T_x^*M$ is the isomorphism induced by the metric.

Ricci curvature. The *Ricci curvature* of (M, g) is obtained by contracting the first covariant variable in R with the only contravariant variable. That is, Ric is a $\binom{2}{0}$-tensor defined by:

$$\forall Y, Z \in \Gamma(TM), \qquad \mathrm{Ric}(Y, Z) = \mathrm{Tr}\,(X \mapsto R(X, Y)Z) = \mathrm{Contr}_1^1(R)(Y, Z). \tag{3}$$

Scalar curvature. The *scalar curvature* S of (M, g) is the trace of Ric seen as a bilinear map on T_xM. That is, the trace of its matrix in any orthonormal basis of T_xM. In order to define it intrinsincally, we first need to use one of the musical isomorphisms defined by the metric in order to get a $\binom{1}{1}$-tensor (i.e. a section of $\mathrm{End}(TM)$) and then take the only possible contraction. Thus,

$$S = \mathrm{Contr}_1^1(\mathrm{Ric}^\sharp) = \mathrm{Tr}(Y \mapsto \mathrm{Ric}(Y, \cdot)^\sharp), \tag{4}$$

where $^\sharp : T_x^*M \to T_xM$ is the isomorphism induced by g and we applied it to one of the variables in Ric (which one not important since Ric and g are symmetric).

Sectional curvature. If two vectors $u, v \in T_xM$ are linearly independent, then the *sectional curvature* of the plane P spanned by u and v is:

$$K(P) = \frac{R(u, v, v, u)}{g(u, u)g(v, v) - g(u, v)^2}. \tag{5}$$

It depends only on P and not on a choice of basis.

2 Symmetries

The Riemann curvature is skew-symmetric in the first two variables and satisfies the first Bianchi identity. That is, for any vector fields X, Y and Z, we have:

$$R(X, Y)Z = -R(Y, X)Z, \tag{6}$$

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0. \tag{7}$$

Its fully-covariant version presents additional symmetries. For any vector fields X, Y, Z and T, we have:

$$\widetilde{R}(X,Y,Z,T) = -\widetilde{R}(Y,X,Z,T) = -\widetilde{R}(X,Y,T,Z) = \widetilde{R}(Z,T,X,Y), \tag{8}$$
$$\widetilde{R}(X,Y,Z,T) + \widetilde{R}(Y,Z,X,T) + \widetilde{R}(Z,X,Y,T) = 0. \tag{9}$$

Finally, the Ricci tensor is symmetric. For any $Y, Z \in \Gamma(TM)$, we have:

$$\mathrm{Ric}(Y,Z) = \mathrm{Ric}(Z,Y). \tag{10}$$

3 Expression in local coordinates

Let $(x_1, \ldots, x_n)$ denote local coordinates on some open subset of M. As usual, we denote by $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ and $(dx^1, \ldots, dx^n)$ the associated local frames of TM and T^*M respectively. The matrix of g in these coordinates is $(g_{ij})_{1\leqslant i,j\leqslant n}$, where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. We denote by $(g^{ij})_{1\leqslant i,j\leqslant n}$ the coefficients of the inverse matrix.

Christoffel symbols. The *Christoffel symbols* $(\Gamma^k_{ij})_{1\leqslant i,j,k\leqslant n}$ of the connection ∇ in these coordinates are defined by the following relations:

$$\forall i,j \in [\![1,n]\!], \qquad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x_k}. \tag{11}$$

Note that, since the Levi–Civita connection is torsion-free, we have $\Gamma^k_{ij} = \Gamma^k_{ji}$ for any i, j, k. The Christoffel symbols are given in local coordinates by:

$$\forall i,j,k \in [\![1,n]\!], \qquad \Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^n g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right). \tag{12}$$

Riemann tensor. Let us define the coefficients $(R^l_{ijk})_{1\leqslant i,j,k,l\leqslant n}$ by:

$$\forall i,j,k \in [\![1,n]\!], \qquad R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_{l=1}^n R^l_{ijk} \frac{\partial}{\partial x_l}. \tag{13}$$

By Eq. (6) and (7), for any i, j, k and l, we have:

$$R^l_{ijk} = -R^l_{jik} \tag{14}$$
$$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0. \tag{15}$$

Then we can write R locally as:

$$R = \sum_{1\leqslant i,j,k,l\leqslant n} R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x_l} = \sum_{\substack{1\leqslant i<j\leqslant n\\ 1\leqslant k,l\leqslant n}} R^l_{ijk}(dx^i \wedge dx^j) \otimes dx^k \otimes \frac{\partial}{\partial x_l}. \tag{16}$$

We proved that for any i, j, k and $l \in [\![1, n]\!]$, we have:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_{m=1}^{r} \Gamma^m_{jk}\Gamma^l_{im} - \sum_{m=1}^{r} \Gamma^m_{ik}\Gamma^l_{jm}. \tag{17}$$

Similarly, we define $(R_{ijkl})_{1\leqslant i,j,k,l\leqslant n}$ by:

$$\forall i, j, k, l \in [\![1, n]\!], \qquad R_{ijkl} = \widetilde{R}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right). \tag{18}$$

By Eq. (8) and (9), for any i, j, k and k, we have:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \tag{19}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0. \tag{20}$$

Then $\widetilde{R}$ can be written locally as:

$$\widetilde{R} = \sum_{1\leqslant i,j,k,l\leqslant n} R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l = \sum_{\substack{1\leqslant i<j\leqslant n\\ 1\leqslant k<l\leqslant n}} R_{ijkl}(dx^i \wedge dx^j) \otimes (dx^k \wedge dx^l). \tag{21}$$

By Eq. (13) and (18), we see that, for any i, j, k and l, we have:

$$R_{ijkl} = \sum_{m=1}^{n} R^m_{ijk} g_{ml} \qquad \text{and} \qquad R^l_{ijk} = \sum_{m=1}^{n} R_{ijkm} g^{ml}. \tag{22}$$

Ricci tensor. As above, we can define the coefficient of Ric by:

$$\forall j, k \in [\![1, n]\!], \qquad \mathrm{Ric}_{jk} = \mathrm{Ric}\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right), \tag{23}$$

so that in local coordinates we have:

$$\mathrm{Ric} = \sum_{1\leqslant j,k\leqslant n} \mathrm{Ric}_{jk}\, dx^j \otimes dx^k. \tag{24}$$

We deduce from the definition of Ric (cf. Eq. (3)) and Eq. (22) that, for any j and k, we have:

$$\mathrm{Ric}_{jk} = \sum_{i=1}^{n} R^i_{ijk} = \sum_{1\leqslant i,m\leqslant n} R_{ijkm} g^{mi}. \tag{25}$$

Scalar curvature. Since $g\left(\frac{\partial}{\partial x_i}, \cdot\right) = \sum_{j=1}^{n} g_{ij} dx^j$ for any $i \in [\![1, n]\!]$, we have:

$$\forall i \in [\![1, n]\!], \left(\frac{\partial}{\partial x_i}\right)^\flat = \sum_{j=1}^{n} g_{ij} dx^j \qquad \text{and} \qquad \forall j \in [\![1, n]\!], \left(dx^j\right)^\sharp = \sum_{i=1}^{n} g^{ji} \frac{\partial}{\partial x_i}. \tag{26}$$

Then we get:

$$\mathrm{Ric}^\sharp = \sum_{1\leqslant j,k\leqslant n} \mathrm{Ric}_{jk}\, dx^j \otimes (dx^k)^\sharp = \sum_{1\leqslant i,j,k\leqslant n} \mathrm{Ric}_{jk}\, g^{ki} dx^j \otimes \frac{\partial}{\partial x_i}, \tag{27}$$

so that

$$S = \sum_{1\leqslant j,k\leqslant n} \mathrm{Ric}_{jk}\, g^{kj}. \tag{28}$$

In nice coordinates. Let us now assume that our coordinates are such that $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is orthonormal at $x = 0$. Then we have $(g_{ij}(0))_{1\leqslant i,j\leqslant n} = I_n = (g^{ij}(0))_{1\leqslant i,j\leqslant n}$, where I_n is the identity matrix of size n. In these coordinates we can simplify Eq. (12), (22), (25) and (28) in the following way:

$$\forall i,j,k \in [\![1,n]\!], \qquad \Gamma_{ij}^k(0) = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j}(0) + \frac{\partial g_{jk}}{\partial x_i}(0) - \frac{\partial g_{ij}}{\partial x_k}(0)\right), \tag{29}$$

$$\forall i,j,k,l \in [\![1,n]\!], \qquad R_{ijk}^l(0) = R_{ijkl}(0), \tag{30}$$

$$\forall j,k \in [\![1,n]\!], \qquad \mathrm{Ric}_{jk}(0) = \sum_{i=1}^n R_{ijk}^i(0) = \sum_{i=1}^n R_{ijki}(0), \tag{31}$$

$$S(0) = \sum_{j=1}^n \mathrm{Ric}_{jj}(0). \tag{32}$$

In normal coordinates. If we assume that $(x_1, \dots, x_n)$ are normal coordinates around the point of coordinates $x = 0$, then we have:

$$(g_{ij}(x))_{1\leqslant i,j\leqslant n} = I_n + O\left(\|x\|^2\right) = (g^{ij}(x))_{1\leqslant i,j\leqslant n}. \tag{33}$$

It is then possible to simplify further Eq. (12) and (17) to get:

$$\forall i,j,k \in [\![1,n]\!], \qquad \Gamma_{ij}^k(0) = 0, \tag{34}$$

$$\forall i,j,k,l \in [\![1,n]\!], \quad R_{ijk}^l(0) = \frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x_i \partial x_k}(0) - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l}(0) - \frac{\partial^2 g_{il}}{\partial x_j \partial x_k}(0) + \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l}(0)\right). \tag{35}$$

4 The case of surfaces

When $n = 2$, we have $\widetilde{R} = R_{1212}(dx^1 \wedge dx^2) \otimes (dx^1 \wedge dx^2)$ for any choice of local coordinates (cf. Eq. (21)). For any $p \in M$, let $\kappa(p) = K(T_pM)$ denote the sectional curvature of the tangent plane. In any local coordinates centered at p and such that $\left(\frac{\partial}{\partial x_1}(0), \frac{\partial}{\partial x_2}(0)\right)$ is orthonormal we have $\kappa(p) = \widetilde{R}\left(\frac{\partial}{\partial x_1}(0), \frac{\partial}{\partial x_2}(0), \frac{\partial}{\partial x_2}(0), \frac{\partial}{\partial x_1}(0)\right) = -R_{1212}(0)$, so that at the point p:

$$\widetilde{R}_p = -\kappa(p)(dx^1 \wedge dx^2) \otimes (dx^1 \wedge dx^2). \tag{36}$$

A computation gives that, for any vector fields X, Y, Z, T on M, we have:

$$\widetilde{R}(X,Y,Z,T) = \kappa\left(g(Y,Z)g(X,T) - g(X,Z)g(Y,T)\right). \tag{37}$$

By definition of $\widetilde{R}$, this means that:

$$R(X,Y)Z = \kappa\left(g(Y,Z)X - g(X,Z)Y\right), \tag{38}$$

for any X, Y and $Z \in \Gamma(TM)$. The definition of Ric as a trace yields:

$$\forall Y, Z \in \Gamma(TM), \qquad \mathrm{Ric}(Y,Z) = \kappa g(Y,Z). \tag{39}$$

Finally, we get:

$$S = 2\kappa. \tag{40}$$