Exterior differential, Stokes Theorem

Exercice 1 (Warm-up)**.** Compute $d\omega$, where ω is the following n-form on $\mathbb{R}^{n+1}$:

$$x \mapsto \sum_{i=0}^{n} (-1)^i x_i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

Exercice 2 (Angle form)**.** Let α be the 1-form $(x, y) \mapsto \dfrac{xdy - ydx}{x^2 + y^2}$ on $\mathbb{R}^2 \setminus \{0\}$ and let $f : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ from $\mathbb{R}_+^* \times \mathbb{R}$ to $\mathbb{R}^2$.

1. Compute $d\alpha$.
2. Is $f^*(\alpha)$ a closed form? Is it exact?
3. Is α exact?
   Hint: consider $i^*\alpha$ where $i : \mathbb{S}^1 \to \mathbb{R}^2$ is the canonical injection and prove that if $i^*\alpha$ was exact, then it would vanish somewhere on $\mathbb{S}^1$.

Exercice 3. Let ω be a volume form on a manifold M. Prove that around any point of M there exist local coordinates $(x_1, \dots, x_n)$ such that $\omega = dx^1 \wedge \cdots \wedge dx^n$.

Exercice 4 (div, rot and all that kind of things)**.**

1. Let $(E, \langle \cdot, \cdot \rangle)$ be an oriented Euclidean space of dimension 3. Let X be a vector field on E, then $\langle X, \cdot \rangle$ defines a 1-form. We define $\mathrm{rot}(X)$ as the only vector field such that $\mathrm{rot}(X) \lrcorner dV = d(\langle X, \cdot \rangle) \in \Omega^2(E)$. Compute the expression of $\mathrm{rot}(X)$ in a direct orthonormal basis of E.
2. Let M be a smooth manifold equipped with a volume form ω. Let X be a vector field on M, we define $\mathrm{div}(X)$ as the only function such that $d(X \lrcorner \omega) = \mathrm{div}(X)\omega$. Compute the expression of $\mathrm{div}(X)$ in local coordinates such that $\omega = dx^1 \wedge \cdots \wedge dx^n$.
3. Prove that $\mathrm{div}(X) \equiv 0$ if and only if the flow of X is volume preserving.

Exercice 5 (Stokes formula)**.**

1. Let M be a compact oriented manifold without boundary and let $\alpha \in \Omega^{n-1}(M)$, compute $\displaystyle\int_M d\alpha$.
2. Is it possible for a volume form on M to be exact?
3. Is it possible on an oriented manifold without boundary that is not compact?