Differential forms, orientability

Exercise 1. 1. Let $f : t \mapsto e^t$ from $\mathbb{R}$ to $\mathbb{R}_+^*$ and let $\alpha = \frac{dx}{x}$, compute $f^*\alpha$.

2. Same question with $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ and $\alpha = dx \wedge dy$.

Let us recall the definition of orientability.

Definition (Orientability)**.** A manifold M of dimension n is said to be *orientable* if there exists a differential form $\omega \in \Omega^n(M)$ such that $\omega_x \neq 0$ for all x in M. That is, for any $x \in M$, if $(v_1(x), ..., v_n(x))$ is a basis of T_xM then $\omega_x(v_1(x), ..., v_n(x)) \neq 0$. Such a form ω is said to be a volume form.

Exercise 2 (Spheres)**.** 1. Let $dV = dx^0 \wedge \cdots \wedge dx^n$ denote the standard volume form on $\mathbb{R}^{n+1}$ (that is the $(n+1)$-form equal to the determinant at each point) and let X be the radial vector field : $x \mapsto \sum x_i \frac{\partial}{\partial x_i}$.
What is $\omega = X \lrcorner dV$ (recall that $(Y \lrcorner \alpha)(Y_1, \ldots, Y_p) = \alpha(Y, Y_1, \ldots, Y_p)$)?

2. Show that ω is invariant under the action of $SO_{n+1}(\mathbb{R})$.

3. Let $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the canonical embedding, Show that $i^*(\omega)$ is a volume form.

Definition (Parallelizable)**.** We say that a manifold M is *parallelizable* if there exist n vector fields $(X_1, \ldots, X_n)$ on M such that, for any $x \in M$, $(X_1(x), \ldots, X_n(x))$ is a basis of T_xM.

Exercise 3 (Orientability)**.** 1. Show that any parallelizable manifold is orientable.

2. Show that a product of two orientable manifolds is itself orientable.

3. Show that the tangent bundle of a manifold is an orientable manifold.

Exercise 4 (Torus)**.** Is the torus $\mathbb{T}^n$ orientable? If so, give an explicit volume form.

Exercise 5 (Projective spaces)**.** 1. Let $n \in \mathbb{N}$ and $f : \mathbb{S}^n \to \mathbb{S}^n$, $x \mapsto -x$. Is this map orientation-preserving?

2. Is the projectif space $\mathbb{RP}^n$ orientable?