Multilinear algebra

Exercise 1 (Tensor products over $\mathbb{R}$)**.** Let E and F be two $\mathbb{R}$-vector spaces of dimension n and m, respectively. We denote by $e = (e_j)$ a basis of E and by $f = (f_i)$ a basis of F.

1. Let $\alpha \in (E^*)^{\otimes k}$, identify the coordinates of α in the basis of $(E^*)^{\otimes k}$ associated with e.
2. Give a natural isomorphism between $E^* \otimes F$ and $\mathcal{L}(E, F)$.
3. Let $L : E \to F$ be a linear map whose matrix is $M = (m^i_j)$ in the bases e and f. We define $L^* : \alpha \mapsto \alpha \circ L$ from F^* to E^*. Give the matrix of L^* in the dual bases e^* and f^*.

Exercise 2 (Exterior product and determinant)**.** Let E be an n-dimensional vector space. We denote by (e_i) a basis of E and by (e^i) its dual basis. Let $k \in \{1, \dots, n\}$ an integer.

1. Let α and β be multilinear forms, check that $\mathrm{Alt}(\alpha \otimes \beta) = \mathrm{Alt}(\mathrm{Alt}(\alpha) \otimes \mathrm{Alt}(\beta))$.
2. Let $\alpha^1, \dots, \alpha^k \in E^*$, show that $\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(v_1, \dots, v_k) = \det\left((\alpha^i(v_j))_{1 \leqslant i,j \leqslant k}\right)$, for any $v_1, \dots, v_k \in E$.
3. For any $I \subset \{1, \dots, n\}$, we denote by $\mathrm{Card}(I)$ its cardinal and by $|I| = \sum_{i \in I} i$ its length. Using the associativity and anticommutativity of $\wedge$ and the previous question, prove the following "Laplace expansion": for any matrix M of dimension n,
$$\det(M) = \sum_{\substack{I,J \subset \{1,\dots,n\} \\ \mathrm{Card}(I) = k = \mathrm{Card}(J)}} (-1)^{|I|+|J|} \det(M_{I,J}) \det(M_{I^C,J^C}),$$
where I^C (resp. J^C) is the complementary set of I (resp. J) and $M_{I,J}$ is the submatrix of M formed by the coefficients which indices lie in $I \times J$. What happens when $k = 1$?

Exercise 3 (Pullback)**.** Let E and F be two vector spaces and $L : E \to F$ be a linear map.

1. For any alternating forms α and β, show that $L^*(\alpha \wedge \beta) = L^*(\alpha) \wedge L^*(\beta)$.
2. Let (e_j) and (f_i) be bases of E and F respectively. We denote by $M = (m^i_j)$ the matrix of L in these bases. Let $J = \{j_1, \dots, j_k\} \subset \{1 \dots, n\}$ be such that $1 \leqslant j_1 < \cdots < j_k \leqslant n$, we denote $e^J = e^{j_1} \wedge \cdots \wedge e^{j_k}$ and use similar notations for F. Let $\omega = \sum \omega_I f^I$, where we sum over subsets $I \subset \{1, \dots, n\}$ of cardinal k. Express $L^*(\omega)$ in the basis (e^J).

Exercise 4 (Exterior algebra)**.** 1. Is there an alternating multilinear form α on a vector space E such that $\alpha \wedge \alpha \neq 0$?

2. Is there a non-zero alternating form commuting with any other?

Exercise 5 (Decomposable forms)**.** Let E be a vector space of dimension n. An alternating k-linear form on E is said to be *decomposable* if it can be written as the alternating product of k linear forms. If not, it is said *indecomposable*.

1. Show that linear forms and alternating n-linear forms are always decomposable.
2. Let $\alpha \in E^* \setminus \{0\}$, show that an alternating k-linear form $\omega \neq 0$ is divisible by α (that is, can be written as $\alpha \wedge \beta$) if and only if $\alpha \wedge \omega = 0$.

3. Let $(\alpha, \beta, \gamma, \delta)$ be linearly independent in E^*. Is the 2-form $\omega = \alpha \wedge \beta + \gamma \wedge \delta$ decomposable?

4. Is an $(n-1)$-form ω always decomposable (assuming $n > 1$)? Consider $\phi_\omega : \alpha \mapsto \alpha \wedge \omega$ from E^* to $\bigwedge^n E^*$.