Submanifolds, partitions of unity

Exercise 1 (Veronese embedding)**.** Recall that $\mathbb{RP}^n$ is defined as $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $\sim$ is the colinearity equivalence relation. If $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we denote by $(x_0 : \cdots : x_n) \in \mathbb{RP}^n$ the line spanned by x. Since $(x_0 : \cdots : x_n) = (\lambda x_0 : \cdots : \lambda x_n)$ for any $\lambda \in \mathbb{R}^*$, these are called homogeneous coordinates.

Let $h : \mathbb{RP}^2 \to \mathbb{RP}^5$ be defined by $(x : y : z) \longmapsto (x^2 : y^2 : z^2 : xy : yz : zx)$, check that h is well-defined and prove that it is an embedding.

Exercise 2 (Global equation)**.** Let M be a smooth compact connected hypersurface in $\mathbb{R}^n$. We admit that $\mathbb{R}^n \setminus M$ has exactly two connected component, one bounded and one unbounded. Prove that there exists a global regular equation for M, that is there exists $f : \mathbb{R}^n \to \mathbb{R}$ smooth, such that $M = f^{-1}(0)$ and $d_x f$ does not vanish on M.

Exercise 3 (Retraction of sublevels)**.** Let M be a compact manifold and $f : M \to \mathbb{R}$ be a smooth function. We denote by M_a the sublevel $\{x \in M \mid f(x) < a\}$. Let $a, b \in \mathbb{R}$ be such that $]a - \varepsilon, b + \varepsilon[$ does not contain any critical value of f, for some positive ε. Prove that M_a and M_b are diffeormorphic.