Tangent spaces, tangent maps

Exercise 1 (Tangent space of a submanifold)**.** Describe the tangent space $T_pM \subset T_pN$ of the submanifold M in N at a point p, for each of the four characterizations of a submanifold.

Exercise 2 (Tangent space of a product)**.** Let M and N be two smooth manifolds, find a natural isomorphism between $T_{(p,q)}(M \times N)$ and $T_pM \times T_qN$.

Exercise 3 (Computation of a differential)**.** Compute the differential of $\overline{F} : \mathbb{T}^2 \to \mathbb{S}^2$ defined as the quotient of the map from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$F : (x, y) \mapsto (\cos(2\pi x)\cos(2\pi y), \cos(2\pi x)\sin(2\pi y), \sin(2\pi x)).$$

On which set is $\overline{F}$ a local diffeomorphism? Is $\overline{F}$ restricted to this domain a global diffeomorphism?

Exercise 4. Is it possible to immerse a compact manifold M of dimension $n > 0$ in $\mathbb{R}^n$?