Manifolds, differentiable maps

Exercise 1 (The sphere)**.**

1. Prove that $\mathbb{S}^n = \left\{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \right\}$ is a smooth manifold of dimension n, by building a smooth atlas.
2. The Euclidean sphere $\mathbb{S}^n$ is also a manifold as a submanifold of $\mathbb{R}^{n+1}$ (see previous exercise sheet, exercise 4). Show that the differentiable structure built in question 1 is the same as the one induced by $\mathbb{R}^{n+1}$.
3. What happens if we replace the Euclidean norm $\|\cdot\|_2$ by another norm $\|\cdot\|$?

Exercise 2 (Product manifolds)**.** Let M and N be smooth manifolds, show that $M \times N$ is a smooth manifold. What is its dimension?

Exercise 3 (The torus)**.**

1. Build a differentiable structure on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that the canonical projection $p : \mathbb{R}^n \to \mathbb{T}^n$ is a local diffeomorphism.
2. In the previous exercise sheet, we built an homeomorphism from $\mathbb{T}^n$ to $(\mathbb{S}^1)^n \subset \mathbb{C}^n$ (exercise 5). Prove that this is in fact a diffeomorphism.

Exercise 4 (The projective space)**.** Recall that $\mathbb{RP}^n$ is defined as $\left(\mathbb{R}^{n+1} \setminus \{0\}\right)/\mathbb{R}^*$, where $\mathbb{R}^*$ acts on $\mathbb{R}^{n+1}$ by dilation. As a set, $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$.

Let $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we denote by $(x_0 : \cdots : x_n)$ its class in $\mathbb{RP}^n$.

1. Let $i \in \{0, \dots, n\}$. Show that $U_i = \{(x_0 : \cdots : x_n) \in \mathbb{RP}^n \mid x_i \neq 0\}$ is open in $\mathbb{RP}^n$, and that it is homeomorphic to $\mathbb{R}^n$ via the canonical projection $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$.
2. Show that $\mathbb{RP}^n$ is a smooth manifold.
3. Show that p is smooth.
4. Show that the restriction of p to $\mathbb{S}^n$ is a local diffeomorphism.