Reminder on topology and calculus

Exercise 1 (Calculus)**.** Show that the following maps are differentiable, and compute their differentials.

1. $(A, B) \longmapsto AB$ from $\mathcal{M}_{nk}(\mathbb{R}) \times \mathcal{M}_{kp}(\mathbb{R})$ to $\mathcal{M}_{np}(\mathbb{R})$.
2. $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.
3. $f \mapsto f^{-1}$ from $GL(E)$ to itself, where E is a real vector space of finite dimension.
4. Let $\Omega \subset \mathbb{R}^n$ be an open set of compact adherence and V a vector space of finite dimension of functions $\mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^1$. We consider $F : (f, x) \mapsto f(x)$ from $V \times \Omega$ to $\mathbb{R}$.

Exercise 2 (Submanifolds)**.** 1. Recall the four equivalent definitions of a d-dimensional submanifold of $\mathbb{R}^n$ (without proving that they are equivalent).

2. Among the following sets, which ones are submanifolds of $\mathbb{R}^n$? Give their dimensions. No precise justification is expected.
   (a) The sphere of radius 1 in $\mathbb{R}^n$ for the euclidean norm.
   (b) The sphere of radius 1 in $\mathbb{R}^n$ for the sup norm.
   (c) A disjoint union of a straight line and a plane in $\mathbb{R}^3$.
   (d) $\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$.
   (e) $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} \mid x^2 - y^2 = 0\}$.
   (f) The image of $h :]-\infty, 1[\to \mathbb{R}^2$ with $h : t \mapsto \left(\frac{t^2-1}{t^2+1}, t\frac{t^2-1}{t^2+1}\right)$.
3. Let Ω be an open subset of $\mathbb{R}^d$ and $h : \Omega \to \mathbb{R}^n$ an injective immersion, is $h(\Omega)$ a submanifold of $\mathbb{R}^n$?

Exercise 3 (Classical groups of matrices)**.** Show that the following subgroups $\mathcal{M}_n(\mathbb{R})$ are submanifolds of $\mathcal{M}_n(\mathbb{R})$, and give their respective dimensions.

1. $GL_n(\mathbb{R})$,
2. $SL_n(\mathbb{R})$ (subgroup of matrices of determinant 1),
3. $O_n(\mathbb{R})$ (subgroup of orthogonal matrices).

Exercise 4 (A submanifold is a manifold)**.** Let Σ be a submanifold of dimension d of $\mathbb{R}^n$ with $d \leqslant n$ and let $\iota : \Sigma \to \mathbb{R}^n$ be the canonical injection from Σ into $\mathbb{R}^n$.

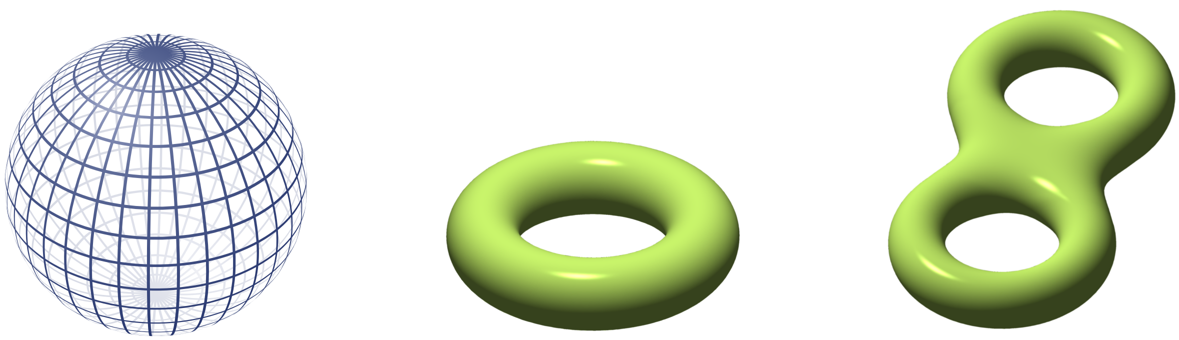
1. Show that Σ is a manifold.
2. Let f be a smooth map from $\mathbb{R}^n$ to $\mathbb{R}$. Show that the restriction $f_{/\Sigma} : \Sigma \to \mathbb{R}$ is smooth.

Exercise 5 (Quotient topology)**.** 1. Let X be a topological space and $\sim$ be an equivalence relation on X. We denote by $p : X \to X/\sim$ the canonical projection. Recall the definition of the quotient topology on $X/\sim$?

2. Let $f : X/\sim \to Y$. Show that f is continuous if and only if $f \circ p$ is.

3. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Show that $\mathbb{T}^n$ is compact and Hausdorff, and that p is an open map.
4. Let $f : K \to Y$ be continuous and bijective with K compact Hausdorff and Y Hausdorff. Show that f is an homeomorphism? Give a counter-example if Y is not Hausdorff.
5. We define $\mathbb{S}^1$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that $\mathbb{T}^1$ is homeomorphic to $\mathbb{S}^1$. More generally, show that $\mathbb{T}^n$ is homeomorphic to $(\mathbb{S}^1)^n \subset \mathbb{C}^n$.
6. Let $\mathbb{RP}^n$ be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation "being colinear". Show that $\mathbb{RP}^n$ is compact Hausdorff and that p is open.

Exercise 6 (Surface of genus g)**.** For any $g \in \mathbb{N}$, find a explicit map $F_g : \mathbb{R}^3 \to \mathbb{R}$ such that $\Sigma_g = (F_g)^{-1}(0)$ is a smooth surface of genus g (that is a torus with g holes).



(a) $g = 0$ (b) $g = 1$ (c) $g = 2$

Hint: Seek F_g of the form $F_g : (x, y, z) \mapsto z^2 - f_g(x, y)$, where $f_g : \mathbb{R}^2 \to \mathbb{R}$ is smooth.